Topological structure of non-contractible loop space and closed geodesics on real projective spaces with odd dimensions

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Abstract

In this paper, we use Chas-Sullivan theory on loop homology and Leray-Serre spectral sequence to investigate the topological structure of the non-contractible component of the free loop space on the real projective spaces with odd dimensions. Then we apply the result to get the resonance identity of non-contractible homologically visible prime closed geodesics on such spaces provided the total number of such distinct closed geodesics is finite.

Key words: Chas-Sullivan theory, Leray-Serre spectral sequence, closed geodesics, real projective spaces, non-simply connected, Morse theory, resonance identity

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1 Introduction and the main results

In this paper, we study the topological structure of the non-contractible component of the free loop space on the real projective spaces with odd dimensions $\mathbb{R}P^{2n+1}$, which are the typically oriented and non-simply connected manifolds with the fundamental group $\pi_1(\mathbb{R}P^{2n+1}) = \mathbb{Z}_2$. Then we

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apply the result to get the resonance identity of non-contractible homologically visible prime closed geodesics on such spaces when the total number of such distinct closed geodesics is finite.

Let (M, F) be a Finsler manifold with finite dimension. As usual we choose an auxiliary Riemannian metric on M (cf. [30]). It endows the free loop space ΛM on M defined by

$$\Lambda M = \left\{ c : S^1 \to M \mid c \text{ is absolutely continuous and } \int_0^1 F(\dot{c}, \dot{c}) dt < +\infty \right\},$$

with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries. This is widely used in geometric analysis. It is also custom to consider the free loop space of continuous closed curves $LM = C^0(S^1, M)$ in topology. Since ΛM and LM are G-weakly homotopy equivalent with any subgroup G of $O(2)$ (cf. Hingston [19], p.101), we shall use both notations and do not distinguish them in this paper.

It is well known (cf. Chapter 1 of [22]) that γ is a closed geodesic or a constant curve on (M, F) if and only if γ is a critical point of the energy functional

$$E(c) = \frac{1}{2} \int_0^1 F(\dot{c}, \dot{c}) dt.$$

For more studies on the closed geodesics, we refer the readers to the survey papers of Bangert [2], Long [24], and Taimanov [31]. Among others, two important topological invariants associated to the free loop space ΛM of a compact manifold M are used by many mathematicians in the study of multiplicity and stability of closed geodesics on M.

The first one is the Betti number sequence $\{\beta_k(\Lambda M; \mathbb{F})\}_{k \in \mathbb{Z}}$ of the free loop space ΛM with $\beta_k(\Lambda M; \mathbb{F}) = \mathrm{rank} H_*(\Lambda M; \mathbb{F})$ for an arbitrary field $\mathbb{F}$. In 1969 Gromoll and Meyer ([16], Theorem 4) established the existence of infinitely many distinct closed geodesics on M, provided that $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ is unbounded. Then Vigué-Poirrier and Sullivan [33] further proved in 1976 that for a compact simply connected manifold M, the Gromoll-Meyer condition holds if and only if $H^*(M; \mathbb{Q})$ is generated by more than one element. Although the Gromoll-Meyer theorem is valid actually for any field $\mathbb{F}$, and there are really some spaces with bounded $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ but unbounded $\{\beta_k(\Lambda M; \mathbb{Z}_2)\}_{k \in \mathbb{Z}}$, it can not be applied to the compact globally symmetric spaces of rank 1 which consist in

$$S^n, \ \mathbb{R}P^n, \ \mathbb{C}P^n, \ \mathbb{H}P^n \text{ and } \mathrm{CaP}^2, \tag{1.1}$$

since $\{\beta_k(\Lambda M, \mathbb{F})\}_{k \in \mathbb{Z}}$ with M in (1.1) is bounded with respect to any field $\mathbb{F}$ (cf. Remark in [36], pp. 17-18 of Ziller).

The second one is the Betti number sequence $\{\bar{\beta}_k(\Lambda M, \Lambda^0 M; \mathbb{F})\}_{k \in \mathbb{Z}}$ of the free loop space pair $(\Lambda M, \Lambda^0 M)$ with $\bar{\beta}_k(\Lambda M, \Lambda^0 M; \mathbb{F})$ being the rank of the relative S^1-equivariant homology

$H_*^{S^1}(\Lambda M, \Lambda^0 M; \mathbb{F})$ defined by

$$H_*^{S^1}(\Lambda M, \Lambda^0 M; \mathbb{F}) = H_*(\Lambda M \times_{S^1} ES^1, \Lambda^0 M \times_{S^1} ES^1; \mathbb{F}), \tag{1.2}$$

where $\Lambda^0 M = M$ is the set of constant curves on M. When $\mathbb{F} = \mathbb{Q}$, it can be proved further that

$$H_*^{S^1}(\Lambda M, \Lambda^0 M; \mathbb{Q}) \cong H_*(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q}), \tag{1.3}$$

which enables Rademacher ([28] of 1989 and [29] of 1992) to establish the resonance identity of prime closed geodesics on the simply connected manifolds of (1.1) provided that their total number is finite. This identity relates local topological invariants of prime closed geodesics to the average Betti number $\bar{B}(\Lambda M, \Lambda^0 M; \mathbb{Q})$ defined by

$$\bar{B}(\Lambda M, \Lambda^0 M; \mathbb{Q}) = \lim_{m\to+\infty} \frac{1}{m} \sum_{k=0}^{m} (-1)^k \bar{\beta}_k(\Lambda M, \Lambda^0 M; \mathbb{Q}), \tag{1.4}$$

and then was used by many authors to study the multiplicity of closed geodesics on such manifolds. One example is the existence of at least two distinct closed geodesics on every 2-dimensional Finsler sphere S^2 proved by Bangert and Long ([5] of 2010).

But for the multiplicity of closed geodesics on non-simply connected manifolds whose free loop space possesses bounded Betti number sequence, we are aware of not many works. For example, in 1981, Ballman, Thorbergsson and Ziller [1] proved that every Riemannian manifold with the fundamental group being a nontrivial finitely cyclic group and possessing a generic metric has infinitely many distinct closed geodesics. In 1984, Bangert and Hingston [3] proved that any Riemannian manifold with fundamental group an infinite cyclic group has infinitely many distinct closed geodesics. To use Morse theory to study this problem, one should know the global topological structure of the free loop space on these manifolds. As far as the authors know, there seems to be only two precise results on real projective spaces obtained by Westerland [34], [35] in the field $\mathbb{Z}_2$.

However when one tries to apply Westerland's results directly to study the multiplicity of closed geodesics on $\mathbb{R}P^d$, two difficulties appear which also lie in the simply connected cases if the coefficient field used is $\mathbb{Z}_2$. The first one is that for every M in (1.1), the following isomorphism

$$H_*^{S^1}(\Lambda M, \Lambda^0 M; \mathbb{Z}_2) \cong H_*(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Z}_2), \tag{1.5}$$

may not hold, which would however be crucial to relate the relative S^1-equivariant homogloy $H_*^{S^1}(\Lambda M, \Lambda^0 M; \mathbb{Z}_2)$ to the $\mathbb{Z}_2$-critical modules of closed geodesics (cf. the definition in (1.13)). Note that such an isomorphism (1.6) is proved in our Lemma 3.3 below for the non-contractible free loop space. The second one is that the $\mathbb{Z}_2$-critical module of even m-iterates of a closed geodesic

c is related to the parity of the difference $i(c^m) - i(c)$ of Morse indices and may lead to unbounded Morse type number sequence so that Gromoll-Meyer type argument is not applicable. We refer readers to Lemma 4.1.4 (ii) on p.127 of [22] as well as Proposition 3.8 and its proof in pp. 345-346 of [5] for more details.

In this paper we overcome the above two difficulties for $\mathbb{R}P^{2n+1}$ by restricting the problem of closed geodesics to the non-contractible component of the free loop space of $\mathbb{R}P^{2n+1}$. More precisely, let $M = \mathbb{R}P^d$ with $d \geq 2$. Then $\pi_1(M) = \mathbb{Z}_2 = \{e, g\}$ with e being the identity and g being the generator of $\mathbb{Z}_2$ satisfying $g^2 = e$ and the free loop space LM possesses a natural decomposition

$$LM = L_e M \bigsqcup L_g M,$$

where $L_e M$ and $L_g M$ are the two connected components of LM whose elements are homotopic to e and g respectively. We shall prove in Lemma 3.3 below

$$H_*^{S^1}(L_g M; \mathbb{Z}_2) \cong H_*(L_g M/S^1; \mathbb{Z}_2). \tag{1.6}$$

Moreover, we observe that every closed geodesic on $\mathbb{R}P^d$ is orientable if and only if $d \in 2\mathbb{N} - 1$, and an m-th iterate c^m of a non-contractible closed geodesic c on $\mathbb{R}P^{2n+1}$ is still non-contractible if and only if m is odd, which then implies $i(c^m) - i(c)$ for odd m is always even. This makes the structure of the $\mathbb{Z}_2$-critical modules of odd iterates of non-contractible closed geodesics be similar to the case in the coefficient field $\mathbb{Q}$ and become rather simple. Thus it is possible to get some information on them, if we can get the topological structure of $L_g\mathbb{R}P^{2n+1}$, which is our first goal in this paper. To this end, we use the following ideas.

(i) Consider the fibrations

$$L_e M \to L_e M \times_{S^1} ES^1 \to BS^1,$$

and

$$L_g M \to L_g M \times_{S^1} ES^1 \to BS^1,$$

as well as their Leray-Serre spectral sequences which we will denote for simplicity by $\mathcal{LS}(L_e M)$ and $\mathcal{LS}(L_g M)$ respectively in this paper.

Define the S^1-equivariant Betti numbers and the Poincaré series of $L_* M$ with $L_* M = LM$, $L_e M$ or $L_g M$ by

$$\bar{\beta}_k(L_* M; \mathbb{Z}_2) = \dim H_k^{S^1}(L_* M; \mathbb{Z}_2), \tag{1.7}$$

$$P^{S^1}(L_* M; \mathbb{Z}_2)(t) = \sum_{k=0}^{\infty} \bar{\beta}_k(L_* M; \mathbb{Z}_2) t^k. \tag{1.8}$$

Here we do not use the relative forms due to the fact that $\Lambda^0 M \cap L_g M = \emptyset$. It then follows

$$P^{S^1}(LM;\mathbb{Z}_2)(t) = P^{S^1}(L_eM;\mathbb{Z}_2)(t) + P^{S^1}(L_gM;\mathbb{Z}_2)(t). \tag{1.9}$$

In Section 4 of this paper, we apply Chas-Sullivan theory and Leray-Serre spectral sequence to carry out related computations to obtain in Theorem 4.1 below four possible Batalin-Vilkovisky (we write B-V below for short) algebraic structures of $\mathbb{H}_*(LM;\mathbb{Z}_2) = H_{*+(2n+1)}(LM;\mathbb{Z}_2)$ according to the behaviors of its generators $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$ constructed in Lemma 4.1.

(ii) In [21] of 1973, Katok constructed a famous family of Finsler metrics N_α for $\alpha \in (0,1) \setminus \mathbb{Q}$ on spheres, specially S^{2n+1}, which possesses precisely $2n+2$ distinct prime closed geodesics. In Section 3 below, via inducing the metrics N_α to $\mathbb{R}P^{2n+1}$, we then prove that the S^1-equivariant Betti number sequence $\{\bar{\beta}_k(L_gM;\mathbb{Z}_2)\}_{k\in\mathbb{Z}}$ is bounded via the boundedness of the Morse number sequence. Such a result will help us simplify the proof of Theorem 1.1.

(iii) In Section 5 of this paper, we compute the S^1-equivariant Poincaré series associated to the third pages of $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ under each of the four possible B-V structures, and then comparing their sums with $P^{S^1}(LM;\mathbb{Z}_2)(t)$ computed by Westerland in [35], we obtain the S^1-equivariant Poincaré series of L_gM, which then yields the average S^1-equivariant Betti number of L_gM defined by

$$\bar{B}(L_gM;\mathbb{Z}_2) = \lim_{m\to+\infty} \frac{1}{m} \sum_{k=0}^{m} (-1)^k \bar{\beta}_k(L_gM;\mathbb{Z}_2). \tag{1.10}$$

The following is our first main result in this paper.

Theorem 1.1 *For $M = \mathbb{R}P^{2n+1}$ with $n \geq 1$, $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ collapse at the second page and the third one respectively. Moreover, the S^1-equivariant Poincaré series of L_gM satisfies*

$$P^{S^1}(L_gM;\mathbb{Z}_2)(t) = \frac{1-t^{2n+2}}{(1-t^{2n})(1-t^2)}, \tag{1.11}$$

and the average S^1-equivariant Betti number of L_gM satisfies

$$\bar{B}(L_gM;\mathbb{Z}_2) = \frac{n+1}{2n}. \tag{1.12}$$

Remark 1.1 *For the complex projective space $\mathbb{C}P^n$, Bökstedt and Ottosen [7] proved in 2007 that $\mathcal{LS}(L\mathbb{C}P^n)$ collapses at the third page.*

Next we apply Theorem 1.1 to study the problem of non-contractible closed geodesics on the Finsler manifold $M = (\mathbb{R}P^{2n+1}, F)$.

Recall that on a Finsler manifold (M, F), for some integer $m \in \mathbb{N}$, the m-th iterate c^m of $c \in \Lambda M$ is defined by

$$c^m(t) = c(mt), \qquad \forall\, t \in [0, 1].$$

The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1 - t)$ for $t \in S^1$.

Definition 1.1 *Let (M, F) be a Finsler manifold (or a Riemannian manifold), and A be a subset of the free loop space ΛM. A closed curve $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ belonging to A is prime in A, if it is not a multiple covering (i.e., iterate) of any other closed curve on M belonging to A. Two prime closed curves c_1 and c_2 in A are distinct (or geometrically distinct), if they do not differ by an S^1-action (or $O(2)$-action). We denote by $\mathrm{CG}(A)$ the set of all closed geodesics on (M, F) (or in Riemannian case) which are distinct prime in A. As usual we denote by $\mathrm{CG}(M, F) = \mathrm{CG}(\Lambda M)$ when $A = \Lambda M$.*

For the Finsler manifold $M = (\mathbb{R}P^{2n+1}, F)$, as in [5], let c be a closed geodesic in M satisfying the following isolated condition:

(ISO) $\qquad S^1 \cdot c^m$ is an isolated critical orbit of E for every $m \geq 1$.

We let

$$\Lambda_g(c) = \{\gamma \in \Lambda_g M \mid E(\gamma) < E(c)\},$$

and define the $\mathbb{Z}_2$-critical module of c^{2m-1} by

$$\bar{C}_q(E, c^{2m-1}; \mathbb{Z}_2) = H_q\left((\Lambda_g(c^{2m-1}) \cup S^1 \cdot c^{2m-1})/S^1, \Lambda_g(c^{2m-1})/S^1; \mathbb{Z}_2\right). \tag{1.13}$$

As we shall prove in Section 3 below, we have

$$\bar{C}_q(E, c^{2m-1}) = H_{q-i(c^{2m-1})}(N^-_{c^{2m-1}} \cup \{c^{2m-1}\}, N^-_{c^{2m-1}}; \mathbb{Z}_2), \tag{1.14}$$

where $N^-_{c^{2m-1}} = N_{c^{2m-1}} \cap \Lambda_g(c^{2m-1})$, $N_{c^{2m-1}}$ is the local characteristic manifold of E at c^{2m-1}, and to get (1.14) we have used the fact $i(c^{2m-1}) - i(c) \in 2\mathbb{Z}$ proved in (3.8) below. Here properties of odd iterates of c are crucial.

As usual, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$ we define the local homological type numbers of c^{2m-1} by

$$k_l(c^{2m-1}) = \dim H_l(N^-_{c^{2m-1}} \cup \{c^{2m-1}\}, N^-_{c^{2m-1}}; \mathbb{Z}_2). \tag{1.15}$$

Based on works of Rademacher in [28], Long and Duan in [26] and [14], we define the *analytical period* n_c of the closed geodesic c by

$$n_c = \min\{j \in 2\mathbb{N} \mid \nu(c^j) = \max_{m \geq 1} \nu(c^m), \ \ \forall\, m \in 2\mathbb{N} - 1\}. \tag{1.16}$$

Note that here in order to simplify the study for non-contractible closed geodesics in $\mathbb{R}P^{2n+1}$, we have slightly modified the definition in [26] and [14] by requiring the analytical period to be even. Then by the same proofs in [26] and [14], we have

$$k_l(c^{2m-1+hn_c}) = k_l(c^{2m-1}), \qquad \forall\, m,\ h \in \mathbb{N},\ l \in \mathbb{Z}. \tag{1.17}$$

For more detailed properties of the analytical period n_c of a closed geodesic c, we refer readers to the two Section 3s in [26] and [14].

As in [4], we have

Definition 1.2 *Let (M, F) be a compact Finsler manifold. A closed geodesic c on M is homologically visible, if there exists an integer $k \in \mathbb{Z}$ such that $\bar{C}_k(E, c) \neq 0$. We denote by $\mathrm{CG}_{\mathrm{hv}}(M, F)$ the set of all distinct homologically visible prime closed geodesics on (M, F).*

It is well known that on a compact Finsler manifold M, there exists at least one homologically visible prime closed geodesic, because the topology of the free loop space ΛM is non-trivial (cf. [2]). Motivated by the resonance identity proved in [28], in Section 6 below, as an application of Theorem 1.1 we obtain the following resonance identity on the non-contractible closed geodesics on Finsler $M = (\mathbb{R}P^{2n+1}, F)$. Note that here if $^{\#}\mathrm{CG}(M) < +\infty$ and c is a prime homologically visible closed geodesic on M, then $\hat{i}(c) > 0$ must hold by Lemma 3.4 below.

Theorem 1.2 *Suppose the Finsler manifold $M = (\mathbb{R}P^{2n+1}, F)$ possesses only finitely many distinct prime closed geodesics, among which we denote the distinct non-contractible homologically visible prime closed geodesics by $c_1, \ldots, c_r$ for some integer $r > 0$. Then we have*

$$\sum_{j=1}^{r} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = \bar{B}(\Lambda_g M; \mathbb{Z}_2) = \frac{n+1}{2n}, \tag{1.18}$$

where the mean Euler number $\hat{\chi}(c_j)$ of c_j is defined by

$$\hat{\chi}(c_j) = \frac{1}{n_j} \sum_{m=1}^{n_j/2} \sum_{l=0}^{4n} (-1)^{l+i(c_j)} k_l(c_j^{2m-1}),$$

and $n_j = n_{c_j}$ is the analytical period of c_j.

Remark 1.2 *(i) Note that homologically invisible closed geodesics, if they exist, have no contributions to the resonance identity (1.18).*

(ii) For the special case when each c_j^{2m-1} is non-degenerate with $1 \leq j \leq r$ and $m \in \mathbb{N}$, we have $n_j = 2$ and $k_l(c_j) = 1$ when $l = 0$, and $k_l(c_j) = 0$ for all other $l \in \mathbb{Z}$. Then (1.18) has the following simple form

$$\sum_{j=1}^{r} (-1)^{i(c_j)} \frac{1}{\hat{i}(c_j)} = \frac{n+1}{n}. \tag{1.19}$$

2 Preliminaries

In this section, we recall some facts on the free loop spaces and Leray-Serre spectral sequence.

2.1 String topology on the loop homology

In their seminar paper [12], Chas and Sullivan introduced a new collection of invariants of manifolds for the loop homology. Specifically, let M be a d-dimensional oriented compact manifold and LM be the free loop space of M. They geometrically constructed the loop product $\bullet$ and the loop bracket $\{,\}$ on the loop homology $\mathbb{H}_*(LM) = H_{*+d}(LM)$, and proved that $(\mathbb{H}_*(LM), \bullet, \{,\})$ is a Gerstenhaber algebra.

Definition 2.1 $(A, \bullet, \{,\})$ *is called a Gerstenhaber algebra, if*

(1) $(A, \bullet)$ *is a graded commutative, associative algebra,*

(2) $\{,\}$ *is a Lie bracket of degree* $+1$, *that is , for every* $a, b, c \in A$,

   (*i*) $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$,

   (*ii*) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}$,

(3) $\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{(|a|+1)|b|} b \bullet \{a, c\}$,

where $|a|$ *denotes the degree of* a.

Consider the circle action $\eta : S^1 \times LM \to LM$, defined by

$$\eta(\theta, \gamma)(t) = \gamma(\theta + t), \ \forall (\theta, \gamma) \in S^1 \times LM.$$

Then, η induces a degree $+1$ operator $\triangle : H_*(LM) \to H_{*+1}(LM)$ defined by

$$\triangle (v) = \eta_*([S^1] \otimes v), \qquad \forall v \in H_*(LM), \tag{2.1}$$

with $[S^1]$ the generator of $H_1(S^1)$. In [12], the authors also proved that $(\mathbb{H}_*(LM), \bullet, \triangle)$ is a Batalin-Vilkovisky algebra.

Definition 2.2 $(A, \bullet, \triangle)$ *is called a Batalin-Vilkovisky algebra, if*

(1) $(A, \bullet)$ *is a graded commutative, associative algebra,*

(2) $\triangle \circ \triangle = 0$,

(3) $(-1)^{|a|} \triangle (a \bullet b) - (-1)^{|a|} \triangle (a) \bullet b - a \bullet \triangle(b)$ *is a derivation of each variable.*

In particular for the free loop space LM, we still have the following B-V formulae

$$\{a, b\} = (-1)^{|a|} \triangle (a \bullet b) - (-1)^{|a|} \triangle (a) \bullet b - a \bullet \triangle(b), \tag{2.2}$$

(see Corollary 5.3 in [12]). Since the coefficient filed used in this paper is $\mathbb{Z}_2$, the B-V formulae (2.2) has the simpler form

$$\triangle (a \bullet b) = \triangle(a) \bullet b + a \bullet \triangle(b) + \{a, b\}, \tag{2.3}$$

which is similar as but a little more complicated than the Leibniz formulae.

In 2002, Cohen and Jones [11] realized Chas-Sullivan loop product $\bullet$ as the cup product $\cup$ in the Hochschild cohomology $HH^*(H^*(M), H^*(M))$, when M is simply connected, that is, they succeeded in establishing a ring isomorphism

$$(\mathbb{H}_*(LM), \bullet) \xrightarrow{\cong} (HH^*(H^*(M), H^*(M)), \cup),$$

(see Theorem 3 in [11] for details). Such a result also holds for $\mathbb{R}P^d$ due to Lemma 5.4 in [34].

2.2 Leray-Serre spectral sequence

The theory of Leray-Serre spectral sequence can be found in many literatures such as Hatcher [17] and McCleary [18]. We sketch it for the reader's convenience.

Consider the fibration

$$F \to E \to B,$$

where F is the connected fiber, E is the total space and the base space B is simply connected. Since the coefficient field in this paper is $\mathbb{Z}_2$, there is a cohomology Leray-Serre spectral sequence $\{E_r^{p,q}, \hat{d}_r\}$ converging to $H^*(E)$. Moreover, the second page

$$E_2^{p,q} \approx H^p(B; H^q(F)) \approx H^p(B) \otimes H^q(F).$$

Similarly, we have the homology Leray-Serre spectra sequence $\{E^r_{p,q}, d_r\}$ converging to $H_*(E)$. Generally speaking, the cohomology form is more powerful than the homology one, since the cohomology groups equipped with the cup product becomes a cohomology ring. By the Leibniz formulae, the differential $\hat{d}_r$ is determined if one has known the value of $\hat{d}_r$ on the generators of the ring.

However, we use in this paper the homology Leray-Serre spectral sequence based on the following observations. Due to the work of [11] and [34], the ring structure $(\mathbb{H}_*(L\mathbb{R}P^{2n+1}; \mathbb{Z}_2), \bullet)$ is clear. Since $(\mathbb{H}_*(L\mathbb{R}P^{2n+1}; \mathbb{Z}_2), \bullet, \triangle)$ is a B-V algebra, the close relationship between the operator $\triangle$ with

the differential d_2 enables us to compute the third page $E^3_{p,q}$ by use of the B-V formulae instead of the Leibniz formulae in the cohomology case.

Let Y be a connected S^1-space with the action map $\eta : S^1 \times Y \to Y$. Consider the composition map

$$Y \xrightarrow{\tau} S^1 \times Y \xrightarrow{\eta} Y,$$

where $\tau(y) = (1, y)$. Since $\eta \circ \tau = id$, the map

$$\eta^* : H^*(Y; \mathbb{Z}_2) \to H^*(S^1 \times Y; \mathbb{Z}_2),$$

is injective. Denote by $\{S^1\}$ the generator of $H^1(S^1; \mathbb{Z}_2)$, then η^* induces a degree -1 map $d : H^*(Y; \mathbb{Z}_2) \to H^{*-1}(Y; \mathbb{Z}_2)$ satisfying that

$$\eta^*(y) = 1 \otimes y + \{S^1\} \otimes dy. \tag{2.4}$$

It is known that d is a linear differential operator, i.e., $d \circ d = 0$, $d(x + y) = dx + dy$ and $d(xy) = xdy + ydx$ (see [6] Proposition 3.2, p. 253).

Lemma 2.1 *([6] Proposition 3.3.) The fibration*

$$Y \to Y \times_{S^1} ES^1 \to BS^1$$

has the following cohomology Leray-Serre spectral sequence:

$$E_2^{*,*} = H^*(BS^1) \otimes H^*(Y) \Rightarrow H^*(Y \times_{S^1} ES^1),$$

The differential in the E_2-page is given by $\hat{d}_2 : H^*(Y) \to \hat{u}H^*(Y)$,

$$\hat{d}_2(1 \otimes y) = \hat{u} \otimes d(y)$$

where d is the differential defined in (2.4) and $\hat{u}$ is the generator of $H^*(BS^1)$ with degree 2.

Remark 2.1 *The above lemma has a more general form:*

$$\hat{d}_2(\hat{u}^p \otimes y) = \hat{u}^{p+1} \otimes d(y), \ \forall\ p \in \mathbb{N} \cup \{0\} \text{ and } y \in H^*(Y).$$

Actually, since $\hat{u}^p \otimes 1$ lies in the horizontal axis of the first quadrant, we have $\hat{d}_2(\hat{u}^p \otimes 1) = 0$ for $p \in \mathbb{N} \cup \{0\}$. By Leibniz formulae, we obtain

$$\begin{aligned}\hat{d}_2(\hat{u}^p \otimes y) &= \hat{d}_2((\hat{u}^p \otimes 1) \cup (1 \otimes y)) \\ &= \hat{d}_2(\hat{u}^p \otimes 1) \cup (1 \otimes y) + (\hat{u}^p \otimes 1) \cup \hat{d}_2(1 \otimes y) \\ &= (\hat{u}^p \otimes 1) \cup \hat{d}_2(1 \otimes y) \\ &= (\hat{u}^p \otimes 1) \cup (\hat{u} \otimes y) \\ &= \hat{u}^{p+1} \otimes y.\end{aligned}$$

In the sequel, we denote $\hat{a} \in H^*(Y)$ the dual of a for every $a \in H_*(Y)$, then $\hat{a}(a) = \langle \hat{a}, a \rangle = 1$. Corresponding to Lemma 2.1, we have

Lemma 2.2 *The fibration $Y \to Y \times_{S^1} ES^1 \to BS^1$ has the following homology Leray-Serre spectral sequence:*

$$E^2_{*,*} = H_*(BS^1) \otimes H_*(Y) \Rightarrow H_*(Y \times_{S^1} ES^1),$$

the differential d_2 in the E^2-page is given by $d_2 : u^p \otimes H_*(Y) \to u^{p-1} \otimes H_*(Y)$

$$d_2(u^p \otimes v) = u^{p-1} \otimes \triangle(v), \ \forall \ p \in \mathbb{N},$$

where $u \in H_2(BS^1)$ is the dual of $\hat{u}$ and $\triangle(v) = \eta_*([S^1] \otimes v)$ with $[S^1]$ the dual of $\{S^1\}$.

Proof: On the one hand, for every $v \in H_{q-1}(Y)$ and $y \in H^q(Y)$ we have from $\eta : S^1 \times Y \longrightarrow Y$ that

$$\begin{aligned}\langle \eta^* y, [S^1] \otimes v \rangle &= \langle 1 \otimes y + \{S^1\} \otimes dy, [S^1] \otimes v \rangle \\ &= \langle \{S^1\} \otimes dy, [S^1] \otimes v \rangle \\ &= \langle dy, v \rangle.\end{aligned}$$

On the other hand,

$$\langle \eta^* y, [S^1] \otimes v \rangle = \langle y, \eta_*([S^1] \otimes v) \rangle = \langle y, \triangle(v) \rangle.$$

So,

$$\langle dy, v \rangle = \langle y, \triangle(v) \rangle. \tag{2.5}$$

Let $w = u^p \otimes v$, then by the duality of cohomology and homology Leray-Serre spectral sequences, Remark 2.1 and (2.5),

$$\begin{aligned}\langle \hat{u}^{p-1} \otimes y, d_2 w \rangle &= \langle \hat{d}_2(\hat{u}^{p-1} \otimes y), w \rangle \\ &= \langle \hat{u}^p \otimes dy, w \rangle \\ &= \langle \hat{u}^p \otimes dy, u^p \otimes v \rangle \\ &= \langle dy, v \rangle \\ &= \langle y, \triangle(v) \rangle \\ &= \langle \hat{u}^{p-1} \otimes y, u^{p-1} \otimes \triangle(v) \rangle,\end{aligned}$$

that is, $d_2(u^p \otimes v) = u^{p-1} \otimes \triangle(v)$. □

Remark 2.2 *In the sequel, we actually do computations in $\{\mathbb{E}^2, d_2\}$, where $\mathbb{E}^2 = H_*(BS^1) \otimes \mathbb{H}_*(Y)$, that is, we only shift the degree of $H_*(Y)$ while do not shift those of $H_*(BS^1)$. Since it is only the change of notation, the above lemma is still valid.*

3 Katok's metrics on spheres and real projective spaces

In this section, we prove that the Betti number sequence of $P^{S^1}(L_g\mathbb{R}P^{2n+1};\mathbb{Z}_2)$ is bounded via Katok's famous metrics on S^{2n+1}. Such a result will help us to simplify the proof of Theorem 1.1.

In 1973, Katok [21] constructed his famous irreversible Finsler metrics on S^n which possess only finitely many distinct prime closed geodesics. His examples were further studied closely by Ziller [37] in 1982, from which we borrow most of the notations for the particular case S^{2n+1}.

Let S^{2n+1} be the standard sphere with the canonical Riemannian metric g and the one-parameter group of isometries

$$\phi_t = \mathrm{diag}(R(pt/p_1), \ldots, R(pt/p_{n+1})),$$

where $p_i \in \mathbb{Z}$, $p = p_1 \cdots p_{n+1}$ and $R(\omega)$ is a rotation in $\mathbb{R}^2$ with angle ω. Let TS^{2n+1} and T^*S^{2n+1} be its tangent bundle and cotangent bundle respectively. Define $H_0, H_1 : T^*S^{2n+1} \to \mathbb{R}$ by

$$H_0(x) = \|x\|_* \quad \text{and} \quad H_1(x) = x(V), \quad \forall x \in T^*S^{2n+1},$$

where $\|\cdot\|_*$ denotes the dual norm of g and V is the vector field generated by ϕ_t. Let

$$H_\alpha = H_0 + \alpha H_1 \qquad \text{for} \quad \alpha \in (0,1).$$

Then $\frac{1}{2}H_\alpha^2$ is homogeneous of degree two and the Legendre transform

$$L_{\frac{1}{2}H_\alpha^2} = D_F\left(\frac{1}{2}H_\alpha^2\right) : T^*S^{2n+1} \to TS^{2n+1},$$

is a global diffeomorphism. Hence,

$$N_\alpha = H_\alpha \circ L^{-1}_{\frac{1}{2}H_\alpha^2}$$

defines a Finsler metric on S^{2n+1}. Since $H_\alpha(-x) \neq H_\alpha(x)$, N_α is not reversible. It was proved that (S^{2n+1}, N_α) with $\alpha \in (0,1) \setminus \mathbb{Q}$ possesses precisely $2(n+1)$ distinct prime closed geodesics (cf. Katok [21] and pp. 137-139 of Ziller [37] for more details).

Consider the antipodal map $A : S^{2n+1} \to S^{2n+1}$ defined by $A(p) = -p$. We first prove

Lemma 3.1 *$A : (S^{2n+1}, N_\alpha) \to (S^{2n+1}, N_\alpha)$ is an isometry.*

Proof: For any $p \in S^{2n+1}$, we consider the following diagram

$$\begin{tikzcd} T_pS^{2n+1} \arrow[r, "A_*"] \arrow[d, "L^{-1}_{\frac{1}{2}H_\alpha^2}"] & T_{-p}S^{2n+1} \arrow[d, "L^{-1}_{\frac{1}{2}H_\alpha^2}"] \\ T_p^*S^{2n+1} \arrow[r, "A^*"] \arrow[dr, "H_\alpha"'] & T_{-p}^*S^{2n+1} \arrow[d, "H_\alpha"] \\ & \mathbb{R} \end{tikzcd} \tag{3.1}$$

Claim 1. *The lower triangle in (3.1) commutes.*

In fact, for any $x \in T_p^* S^{2n+1}$, we have $A^*(x) = -x \in T_{-p}^* S^{2n+1}$ and

$$\begin{aligned} H_\alpha \circ A^*(x) &= H_\alpha(-x) \\ &= \|-x\|_* + (-x)(V_{-p}) \\ &= \|x\|_* + (-x)(-V_p) \\ &= \|x\|_* + x(V_p) \\ &= H_\alpha(x), \end{aligned} \tag{3.2}$$

where the third identity is due to the fact that A is isomorphic for the canonical metric $\|\cdot\|_*$ and $V_{-p} = -V_p$ which follows by the definition of V. This proves the claim.

To prove that A is an isometry of S^{2n+1}, i.e.,

$$H_\alpha \circ L^{-1}_{\frac{1}{2}H_\alpha^2} \circ A_*(X) = H_\alpha \circ L^{-1}_{\frac{1}{2}H_\alpha^2}(X),\ \forall X \in T_p S^{2n+1}, \tag{3.3}$$

it is sufficient by (3.2) to prove

Claim 2. *The upper square in (3.1) commutes, i.e.,*

$$L^{-1}_{\frac{1}{2}H_\alpha^2} \circ A_*(X) = A^* \circ L^{-1}_{\frac{1}{2}H_\alpha^2}(X),\ \forall X \in T_p S^{2n+1}, \tag{3.4}$$

In fact, (3.4) is equivalent to

$$A_* \circ L_{\frac{1}{2}H_\alpha^2}(x) = L_{\frac{1}{2}H_\alpha^2} \circ A^*(x),\ \forall x \in T_p^* S^{2n+1}. \tag{3.5}$$

Note that for any $x \in T_p^* S^{2n+1}$,

$$\begin{aligned} L_{\frac{1}{2}H_\alpha^2}(x) &= D_F\left(\frac{1}{2}H_\alpha^2\right)(x) \\ &= H_\alpha(x) \cdot D_F H_\alpha(x) \\ &= (\|\bar{x}\| + \alpha\langle V_p, \bar{x}\rangle)\left(\frac{\bar{x}}{\|\bar{x}\|} + \alpha V_p\right), \end{aligned}$$

where $\bar{x}$ is the canonical identification of x (cf. p. 143 in [37]). Thus we get

$$\begin{aligned} L_{\frac{1}{2}H_\alpha^2} \circ A^*(x) &= L_{\frac{1}{2}H_\alpha^2}(-x) \\ &= (\|\overline{(-x)}\| + \alpha\langle V_{-p}, \overline{(-x)}\rangle)\left(\frac{\overline{(-x)}}{\|\overline{(-x)}\|} + \alpha V_{-p}\right) \\ &= (\|-\bar{x}\| + \alpha\langle -V_p, -\bar{x}\rangle)\left(\frac{-\bar{x}}{\|-\bar{x}\|} + \alpha(-V_p)\right) \\ &= -(\|\bar{x}\| + \alpha\langle V_p, \bar{x}\rangle)\left(\frac{\bar{x}}{\|\bar{x}\|} + \alpha V_p\right) \\ &= A_* \circ L_{\frac{1}{2}H_\alpha^2}(x), \end{aligned}$$

where the second identity is due to $A^*(x) = -x \in T^*_{-p}S^{2n+1}$. Then (3.5), and then Claim 2 as well as Lemma 3.1 are proved. $\square$

Here note that $M = (\mathbb{R}P^{2n+1}, F)$ is orientable as well-known. Specially we have

Lemma 3.2 *Let $M = (\mathbb{R}P^{2n+1}, F)$ be a Finsler manifold. Then every prime closed geodesic on M is orientable.*

Proof: Note that if γ is a closed geodesic in the homotopy class $\Lambda_e M$ of closed curves on M represented by e in $\mathbb{Z}_2$, it is contractible in M, and thus is orientable.

If γ is a closed geodesic in the homotopy class $\Lambda_g M$ of closed curves on M represented by g in $\mathbb{Z}_2$. Note that M can be identified with the union of the open unit ball

$$D^{2n+1} = \{x = (x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} \mid |x| < 1\}$$

and the quotient space of the boundary S^{2n} of D^{2n+1} module the action of the antipodal map A equipped with the Finsler metric F, i.e.,

$$M = D^{2n+1} \cup (S^{2n}/A). \tag{3.6}$$

Thus this prime closed geodesic γ in $\Lambda_g M$ can be viewed as a curve in D^{2n+1} with the two end points being the two antipodal points p and $-p$ on S^{2n}, and γ runs from $\gamma(0) = p$ to $\gamma(\tau) = -p$. Here γ can not be approximated by closed curves in the closure of D^{2n+1} and is not contractible. But the tangent space $T_p S^{2n}$ is carried to $T_{-p}S^{2n}$ by γ. Therefore γ is orientable. $\square$.

Remark 3.1 *Note that let γ be a closed geodesic on the Finsler manifold $M = (\mathbb{R}P^{2n}, F)$ belonging to $\Lambda_g M$. Then γ is not orientable.*

By Lemma 3.1, we can endow $\mathbb{R}P^{2n+1}$ a Finsler metric induced by N_α, which is still denoted by N_α for simplicity. Therefore the natural projection

$$\pi : (S^{2n+1}, N_\alpha) \to (\mathbb{R}P^{2n+1}, N_\alpha),$$

is locally isometric.

Proposition 3.1 *For $M = (\mathbb{R}P^{2n+1}, N_\alpha)$ with $\alpha \in (0,1) \setminus \mathbb{Q}$ and $n \geq 1$,*

$${}^{\#}\mathrm{CG}(\Lambda_g M) \leq {}^{\#}\mathrm{CG}(\Lambda_e M) \leq {}^{\#}\mathrm{CG}(\Lambda(S^{2n+1}, N_\alpha)) = 2(n+1),$$

where ${}^{\#}CG(A)$ is the number of $CG(A)$ defined in Definition 1.1, and (S^{2n+1}, N_α) denotes the sphere S^{2n+1} endowed with the Katok metric N_α.

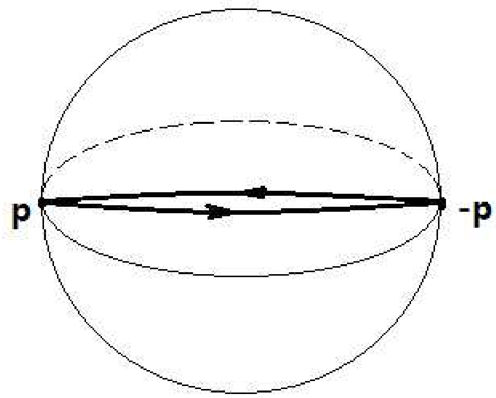

Figure 3.1: The orientability of γ and contractibility of γ^2.

Proof: Note that by Lemma 3.2, every closed geodesic on M is orientable.

For every prime closed geodesic $\gamma \in \Lambda_g M$, the second iterate γ^2 of γ defined by $\gamma^2(t) = \gamma(2t)$ is a prime closed geodesic in $\Lambda_e M$ by the fact $g^2 = e$ in $\mathbb{Z}_2$. This fact can be intuitively understood as follows. As in the proof of Lemma 3.2, M can be identified as in (3.6). Thus a prime closed geodesic γ on M which belongs to $\Lambda_g M$ can be viewed as a curve in D^{2n+1} with the two end points being the two antipodal points p and $-p$ on S^{2n}. Such a closed curve in M can not be approximated by closed curves in the open ball D^{2n+1} and is not contractible. Then γ^2 runs from p to $-p$ along γ first and then runs from $-p$ to p along γ^{-1}. Now this forms a closed curve in the closure $\overline{D^{2n+1}}$ of D^{2n+1}, and thus we can shrink this curve slightly at p and $-p$ into D^{2n+1} to form a new slightly shorter closed curve in D^{2n+1}, and then continue to shrink it to the origin. We refer to Figure 3.1 for an illustration on this argument, where for clarity, we have drawn γ and γ^{-1} slightly away from each other. From this argument, the first inequality claimed by the proposition follows.

Let $\beta : [0, 1] \to M$ be a prime closed geodesic in $\Lambda_e M$ with $\beta(0) = \beta(1) = p$ and $\beta'(0) = \beta'(1)$. Since (S^{2n+1}, N_α) is complete, we get by Hopf-Rinow theorem there exists $\gamma : [0, +\infty) \to S^{2n+1}$ satisfying $\gamma(0) = p$ and $\gamma'(0) = \beta'(0)$. Let $\tilde{\beta} = \pi \circ \gamma$. Since $\pi : (S^{2n+1}, N_\alpha) \to (\mathbb{R}P^{2n+1}, N_\alpha)$ is locally isometric, $\tilde{\beta}$ is a geodesic in M satisfying that $\tilde{\beta}(0) = \beta(0)$ and $\tilde{\beta}'(0) = \beta'(0)$. By the uniqueness of the geodesic, $\tilde{\beta}|_{[0,1]} = \beta$. Since the pre-image of every closed curve in $\Lambda_e M$ is also a closed one in S^{2n+1} and $\beta'(0) = \beta'(1)$ implies $\gamma'(0) = \gamma'(1)$, $\gamma = \gamma|_{[0,1]}$ is a closed geodesic in S^{2n+1}. Note that here γ is prime in ΛS^{2n+1} provided β is prime in $\Lambda_e M$. Thus the second inequality follows. $\square$

Remark 3.2 *(i) Here if γ is a prime closed geodesic in (M, F) belonging to* $\mathrm{CG}(\Lambda_g M)$*, then* $\gamma^2 \in \mathrm{CG}(\Lambda_e M)$ *is also prime in* $\mathrm{CG}(\Lambda_e M)$ *according to the definition 1.1. Note that* $\mathrm{CG}(\Lambda_e M)$ *contains two kinds of closed geodesics: the ones which are two iterates of γ in* $\mathrm{CG}(\Lambda_g M)$ *and the ones which are not so.*

(ii) Although Proposition 3.1 is enough for our arguments in the sequel, we point out that the inequalities therein are actually equalities. Since the closed geodesics found in [37] are great circles of S^{2n+1} and the projection map π is locally isometric, their images under π are exactly the two iterations of the associated non-contractible closed geodesics on $\mathbb{R}P^{2n+1}$ which implies in turn that

$$^{\#}\mathrm{CG}(\Lambda_g M) \geq {}^{\#}\mathrm{CG}(\Lambda S^{2n+1})$$

and so they are equal.

Lemma 3.3 *For $M = \mathbb{R}P^d$ with $d \geq 2$, let $\Lambda_g M$ be the component of ΛM whose elements are homotopic to g. Then,* $H_*^{S^1}(\Lambda_g M; \mathbb{Z}_2) \cong H_*(\Lambda_g M/S^1; \mathbb{Z}_2)$.

Proof: Let $\Lambda[q]$ be the subspace of $\Lambda_g M$ with multiplicity q, then we have

$$\Lambda_g M = \bigcup_{q \in 2\mathbb{N}-1} \Lambda[q].$$

Consider the projection map $\pi : \Lambda_g M \times_{S^1} ES^1 \to \Lambda_g M/S^1$ defined by

$$\pi(l, e) = [l], \ \forall (l, e) \in \Lambda_g M \times_{S^1} ES^1,$$

where $[l]$ is the equivalent class of l under the natural action of S^1.

If $(l, e) \in \Lambda[q] \times_{S^1} ES^1$, then

$$(l, e) = \left(l \cdot \frac{k}{q}, \frac{k}{q} \cdot e\right) = \left(l, \frac{k}{q} \cdot e\right), \ \forall\ 0 \leq k \leq q - 1.$$

Therefore, for every $[x] \in \Lambda_g M/S^1$ with the multiplicity of $q \in 2\mathbb{N} - 1$, we have

$$\pi^{-1}([x]) \simeq ES^1/\mathbb{Z}_q.$$

Since q is odd, we obtain by Theorem III 2.4 in Bredon [9] that

$$H_k(\pi^{-1}([x]); \mathbb{Z}_2) = H_k(ES^1/\mathbb{Z}_q; \mathbb{Z}_2) \cong H_k(ES^1; \mathbb{Z}_2)^{\mathbb{Z}_q} = \begin{cases} \mathbb{Z}_2, & k = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.7}$$

So for the map $\pi : \Lambda_g M \times_{S^1} ES^1 \to \Lambda_g M/S^1$, there exists a Leray spectral sequence $\{E^r, d_r\}$ converging to $H_*(\Lambda_g M \times_{S^1} ES^1; \mathbb{Z}_2)$ with

$$E^2_{s,t} = H_s\left(\Lambda_g M/S^1; \bigcup_{[x]\in\Lambda_g M/S^1} H_t(\pi^{-1}([x]); \mathbb{Z}_2)\right),$$

where $\bigcup_{[x]\in\Lambda_g M/S^1} H_t(\pi^{-1}([x]); \mathbb{Z}_2)$ is the coefficient sheaf on $\Lambda_g M/S^1$ with germ $H_t(\pi^{-1}([x]); \mathbb{Z}_2))$ (cf. [8], p. 35; or [20], p. 37).

By (3.7), the elements of E^2 are all zeroes except for the bottom line and so the spectral sequence $\{E^r, d_r\}$ collapses at E^2. As a result, we get that $H^{S^1}_*(\Lambda_g M; \mathbb{Z}_2) \cong H_*(\Lambda_g M/S^1; \mathbb{Z}_2)$. $\square$

From now on, we always assume that every closed geodesic $c \in \Lambda_g M$ satisfies the isolated condition (ISO) in Section 1. Note that by Lemma 3.2, such a geodesic c is orientable, and its odd iterates stay still in $\Lambda_g M$ and even iterates stay in $\Lambda_e M$. By Theorem 1.1 of [23], the Bott-type iteration formula Theorem 9.2.1 on p.199 and Lemma 5.3.1 on p.120 of [25] can be applied to every closed geodesic $c \in \Lambda_g M$ to get

$$i(c^{2m-1}) = \sum_{\omega^{2m-1}=1} i_\omega(c) = i(c) + 2\sum_{0<k\le m} i_{e^{k\pi/(2m-1)}}(c) = i(c) \mod 2, \tag{3.8}$$

for all $m \in \mathbb{N}$. We also mention that the iterated index formulaes under the rational coefficient field $\mathbb{Q}$ in Section 3 of [5] still hold for $\mathbb{Z}_2$ in this paper. Indeed, the rational coefficient field $\mathbb{Q}$ is crucially used in the proof of Lemma 3.6 in [5] to ensure Theorem III 7.2 of Bredon [9] which also holds for the coefficient field $\mathbb{Z}_2$ since the multiplicity of every closed curve in $\Lambda_g M$ is odd.

Lemma 3.4 *Let (M, F) be a compact Finsler manifold with finite fundamental group and possess only finitely many distinct prime closed geodesics, among which the homologically visible ones are denoted by c_i for $1 \le i \le k$. Then we have*

$$\hat{i}(c_i) > 0, \qquad \forall\, 1 \le i \le k. \tag{3.9}$$

Proof: It is well known that every closed geodesic c on M must have mean index $\hat{i}(c) \ge 0$.

Assume by contradiction that there is a homologically visible closed geodesic c on M satisfying $\hat{i}(c) = 0$. Then $i(c^m) = 0$ for all $m \in \mathbb{N}$ by Bott iteration formula and c must be an absolute minimum of E in its free homotopy class, since otherwise there would exist infinitely many distinct closed geodesics on M by Theorem 3 on p.385 of [4]. It also follows that this homotopy class must be non-trivial.

On the other hand, by Lemma 7.1 of [29], there exists an integer $k(c) > 0$ such that $\nu(c^{m+k(c)}) = \nu(c^m)$ for all $m \in \mathbb{N}$. Specially we obtain $\nu(c^{mk(c)+1}) = \nu(c)$ for all $m \in \mathbb{N}$ and then elements of

$\ker(E''(c^{mk(c)+1}))$ are precisely $mk(c)+1$st iterates of elements of $\ker(E''(c))$. Thus by the Gromoll-Meyer theorem in [15], the behavior of the restriction of E to $\ker(E''(c^{mk(c)+1}))$ is the same as that of the restriction of E to $\ker(E''(c))$. Then together with the fact $i(c^m) = 0$ for all $m \in \mathbb{N}$, we obtain that $c^{mk(c)+1}$ is a local minimum of E in its free homotopy class for every $m \in \mathbb{N}$. Because M is compact and possessing finite fundamental group, there must exist infinitely many distinct closed geodesics on M by Corollary 2 on p.386 of [4]. Then it yields a contradiction and proves (3.9). $\square$.

By Lemmas 3.2 and 3.4, we have the following result.

Proposition 3.2 *For $M = \mathbb{R}P^{2n+1}$ with $n \geq 1$, the S^1-equivariant Betti number sequence $\{\bar{\beta}_k(\Lambda_g M;\mathbb{Z}_2)\}$ of $H_*^{S^1}(\Lambda_g M;\mathbb{Z}_2)$ is bounded.*

Proof: Since the S^1-equivariant Betti numbers $\bar{\beta}_k(\Lambda_g M;\mathbb{Z}_2)$ are topological invariants of M, they are independent of the choice of the Finsler metric F on it. To estimate them, it suffices to choose a special Finsler metric $F = N_\alpha$ for $\alpha \in (0,1) \setminus \mathbb{Q}$, i.e., the Katok metrics. Then for $M = (\mathbb{R}P^{2n+1}, N_\alpha)$ by Proposition 3.1, $\Lambda_g M$ has only finitely many distinct prime non-contractible closed geodesics, among which we denote the homologically visible ones by $c_1, \cdots, c_r$ for some integer $r \in \mathbb{N}$. Note that by Lemma 3.4, each c_j must have positive mean index $\hat{i}(c_j) > 0$ for $1 \leq j \leq r$. Note also that homologically invisible closed geodesics make no contributions to any Morse type numbers. Here for $k \in \mathbb{Z}$, the k-th Morse type number $M_k(\Lambda_g M)$ of $\Lambda_g M$ is defined by

$$M_k(\Lambda_g M) = \sum_{j=1}^{r} \sum_{m\in\mathbb{N}} \dim \bar{C}_k(E, c_j^{2m-1}). \tag{3.10}$$

Now it is well known that the following inequalities hold for some constant $B > 0$,

$$\bar{\beta}_k(\Lambda_g M) \leq M_k(\Lambda_g M) \leq B, \qquad \forall\, k \in \mathbb{Z}, \tag{3.11}$$

where the first inequality follows from the classical Mores theory, and the second inequality follows from the same argument in the proof of Theorem 4 of [16] for homologically visible prime closed geodesics $c_1, \cdots, c_r$, and the proof is complete. $\square$

4 Possible B-V algebraic structures of $(\mathbb{H}_*(L\mathbb{R}P^{2n+1};\mathbb{Z}_2), \bullet, \triangle)$

Let $M = \mathbb{R}P^d$ and $\pi_1(M) = \mathbb{Z}_2 = \{e, g\}$ with the generator g satisfying $g^2 = e$. Then the free loop space LM decomposes into

$$LM = L_e M \sqcup L_g M,$$

with L_eM and L_gM being the connected components of LM whose elements are homotopic to e and g respectively. As in p.9 of [12], we use the following convention: $\mathbb{H}_*(LM;\mathbb{Z}_2) = H_{*+d}(LM;\mathbb{Z}_2)$.

In [34] and later in [35], Westerland obtained the following results.

Proposition 4.1 *For $M = \mathbb{R}P^{2n+1}$ with $n \geq 1$,*

$$\mathbb{H}_*(LM;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,v,w]/(x^{2n+2}, v^2 - (n+1)wx^{2n}),$$

where the topological degree of x, v and w are -1, 0 and $2n$ respectively, i.e., $|x| = -1$, $|v| = 0$ and $|w| = 2n$. Moreover,

$$\mathbb{H}_*(L_eM;\mathbb{Z}_2) \cong \mathbb{H}_*(L_gM;\mathbb{Z}_2). \tag{4.1}$$

Proof: For readers convenience, here we briefly indicate why (4.1) should be true. From the proof of Lemma 5.4 in [34], the Eilenberg-Moore spectral sequence converging to $H^*(LM;\mathbb{Z}_2)$ collapses at the second page, that is, the two Eilenberg-Moore spectral sequences converging to $H^*(L_eM;\mathbb{Z}_2)$ and $H^*(L_gM;\mathbb{Z}_2)$ respectively collapse at the second pages which are the same due to (1) of Corollary 5.3 (alternatively, (1) of Lemma 5.2) therein. As a result, we get

$$H^*(L_eM;\mathbb{Z}_2) \cong H^*(L_gM;\mathbb{Z}_2)$$

and so (4.1) follows. □

Note that it is not obvious where the homology of the two components of the free loop space fit into the above presentation of the homology of LM.

Proposition 4.2 *For $M = \mathbb{R}P^{2n+1}$ with $n \geq 1$, the Poincaré series of $H_*^{S^1}(LM;\mathbb{Z}_2)$ is*

$$\frac{1-t^{2n+2}}{(1-t^{2n})(1-t^2)}\left(1 + \frac{1+t}{1-t^2}\right).$$

Proof: See Theorem 1.1 (2) in [35]. □

We need to extract from this Poincaré series the part coming from the non-contractible component. For this purpose, we first analyze the ring structure of $(\mathbb{H}_*(LM;\mathbb{Z}_2), \bullet)$.

Lemma 4.1 *With the same notations in Proposition 4.1, there exist*

$$\tilde{x} = x + a_1xv + a_2x^{2n+1}w + a_3x^{2n+1}vw \in \mathbb{H}_{-1}(M;\mathbb{Z}_2) \subseteq \mathbb{H}_{-1}(L_eM;\mathbb{Z}_2),$$
$$\tilde{v} = v + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}vw \in \mathbb{H}_0(L_gM;\mathbb{Z}_2),$$

and

$$\tilde{w} = c_0w + c_1\tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \in \mathbb{H}_{2n}(L_eM;\mathbb{Z}_2) \cup \mathbb{H}_{2n}(L_gM;\mathbb{Z}_2),$$

where $a_j, b_j, c_j \in \mathbb{Z}_2$ are some coefficients found in the subsequent proofs of this lemma, such that

$$\mathbb{H}_*(LM;\mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{x},\tilde{v},\tilde{w}]/(\tilde{x}^{2n+2}, \tilde{v}^2 + b_1 - (n+1)\sigma(b_1,c_1)\tilde{x}^{2n}\tilde{v}^{b_1c_1}\tilde{w}), \tag{4.2}$$

where

$$\sigma(b_1,c_1) = \begin{cases} 0, & \textit{if } b_1 = 0 \textit{ and } c_1 = 1; \\ 1, & \textit{otherwise.} \end{cases}$$

Proof: We construct $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$ one by one as follows.

Step 1: Construction of $\tilde{x}$.

Consider first the composed maps

$$M \xrightarrow{i} L_eM \xrightarrow{ev} M,$$

where i is the inclusion map which maps each point in M to the corresponding constant loop in L_eM and $ev(\gamma) = \gamma(0)$ is the evaluation map at the initial point of the curve. Since $ev \circ i = id$, the map

$$i_* : \mathbb{H}_*(M;\mathbb{Z}_2) \to \mathbb{H}_*(L_eM;\mathbb{Z}_2),$$

is an embedding.

Let $\tilde{x} \in \mathbb{H}_{-1}(M;\mathbb{Z}_2)$ be the generator of $\mathbb{H}_*(M;\mathbb{Z}_2)$. It follows by Proposition 4.1 that there exist $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$ such that

$$\tilde{x} = a_0x + a_1xv + a_2x^{2n+1}w + a_3x^{2n+1}vw.$$

We claim $a_0 = 1$, that is,

$$\tilde{x} = x + a_1xv + a_2x^{2n+1}w + a_3x^{2n+1}vw. \tag{4.3}$$

If not, then $\tilde{x} = a_1xv + a_2x^{2n+1}w + a_3x^{2n+1}vw$. By Proposition 4.1, $x^{2n+2} = 0$ and $v^2 = (n+1)x^{2n}w$ which imply

$$\tilde{x}^2 = x^2(a_1v + a_2x^{2n}w + a_3x^{2n}vw)^2 = 0,$$

but this contradicts to $\mathbb{H}_*(M;\mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{x}]/(\tilde{x}^{2n+2})$.

Further direct computations show that for $k \geq 2$,

$$\begin{aligned} \tilde{x}^k &= (x + a_1xv + a_2x^{2n+1}w + a_3x^{2n+1}vw)^k \\ &= x^k(1 + a_1v + a_2x^{2n}w + a_3x^{2n}vw)^k \\ &= x^k(1 + a_1v)^k \\ &= x^k(1 + ka_1v), \end{aligned}$$

where the third identity is due to $x^{2n+k} = 0$ and the forth one follows by $x^k v^2 = (n+1)x^{2n+k}w = 0$. Therefore we obtain that for $k \geq 2$,

$$\tilde{x}^k = \begin{cases} x^k, & \text{if } k \text{ is even or } a_1 = 0, \\ x^k(1+v), & \text{if } k \text{ is odd and } a_1 = 1, \end{cases} \tag{4.4}$$

which implies that

$$\tilde{x}^{2n+2} = 0 \text{ and } v^2 - (n+1)w\tilde{x}^{2n} = 0. \tag{4.5}$$

We come back to consider the case of $k = 1$. Multiplying both sides of (4.3) by v, we obtain

$$\begin{aligned} \tilde{x}v &= xv + a_1 xv^2 + a_2 x^{2n+1}vw \\ &= xv + (n+1)a_1 x^{2n+1}w + a_2 x^{2n+1}vw, \end{aligned}$$

which is equivalent to

$$xv = \tilde{x}v + (n+1)a_1 x^{2n+1}w + a_2 x^{2n+1}vw. \tag{4.6}$$

By direct computations we then get

$$x = \tilde{x} + \tilde{a}_1\tilde{x}v + \tilde{a}_2\tilde{x}^{2n+1}w + \tilde{a}_3\tilde{x}^{2n+1}vw, \tag{4.7}$$

where $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3 \in \mathbb{Z}_2$ can be determined from a_1, a_2 and a_3 via (4.3), (4.4) and (4.6). That is, x can be represented in terms of $\tilde{x}$, v and w.

Due to (4.7) and (4.5), the Chas-Sullivan ring $\mathbb{Z}_2[x, v, w]/(x^{2n+2}, v^2 - (n+1)wx^{2n})$ can also be represented in terms of $\tilde{x}$, v, w, and elements in the set

$$K = \{\tilde{x}^a v^b w^c \mid 0 \leq a \leq 2n+1,\ 0 \leq b \leq 1 \text{ and } 0 \leq c < +\infty\},$$

where elements in K from degree $-(2n+1)$ to 0 can be listed as follows:

$$\begin{cases} \deg = 0; & 1,\ v,\ \tilde{x}^{2n}w,\ \tilde{x}^{2n}vw; \\ \deg = -1; & \tilde{x},\ \tilde{x}v,\ \tilde{x}^{2n+1}w,\ \tilde{x}^{2n+1}vw; \\ \deg = -2; & \tilde{x}^2,\ \tilde{x}^2 v; \\ \deg = -3; & \tilde{x}^3,\ \tilde{x}^3 v; \\ \quad\vdots & \quad\vdots \\ \deg = -2n+1; & \tilde{x}^{2n-1},\ \tilde{x}^{2n-1}v; \\ \deg = -2n; & \tilde{x}^{2n},\ \tilde{x}^{2n}v; \\ \deg = -(2n+1); & \tilde{x}^{2n+1},\ \tilde{x}^{2n+1}v; \end{cases} \tag{4.8}$$

while the elements in K from degree 1 to $2n$ are the corresponding ones from degree $-2n+1$ to 0 in (4.8) multiplying by w; the elements in K with higher degrees can be obtained by the similar

way via multiplying w^k for integers $k > 1$. This representation is an isomorphism by the following claim.

Claim: *The Chas-Sullivan ring $\mathbb{H}_*(LM;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,v,w]/(x^{2n+2}, v^2-(n+1)wx^{2n})$ can be represented isomorphically in terms of $\tilde{x}$, v, and w as*

$$\mathbb{H}_*(LM;\mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{x},v,w]/(\tilde{x}^{2n+2}, v^2-(n+1)\tilde{x}^{2n}w). \tag{4.9}$$

To prove this claim, it suffices to show that the elements in K with the same degree are linearly independent.

Notice also that $yw = 0$ implies $y = 0$ for any $y \in \mathbb{Z}_2[x,v,w]/(x^{2n+2}, v^2-(n+1)wx^{2n})$. Indeed, assume without loss of generality $y = \sum_{1\le i\le i_0} k_i x^{a_i} v^{b_i} w^{c_i}$ with i_0 the dimension of the linear space with degree $|y|$, $k_i \in \mathbb{Z}_2$ and $-a_i + 2nc_i = |y|$. Then, we get

$$\sum_{1\le i\le i_0} k_i x^{a_i} v^{b_i} w^{c_i+1} = yw = 0.$$

But the elements $x^{a_i} v^{b_i} w^{c_i+1}$ with $1 \le i \le i_0$ are linearly independent due to the ring structure itself, which implies $k_i = 0$ for every $1 \le i \le i_0$ and thus $y = 0$.

As a result, we need only to prove that the elements with the same degree listed in (4.8) are linearly independent.

1° For deg $= -1$, assume that there is $(l_1,l_2,l_3,l_4) \in \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ such that

$$l_1\tilde{x} + l_2\tilde{x}v + l_3\tilde{x}^{2n+1}w + l_4\tilde{x}^{2n+1}vw = 0. \tag{4.10}$$

Multiplying both sides of (4.10) with x, we get by (4.3) and (4.4) that

$$0 = l_1x(x+a_1xv) + l_2x^2v = l_1x^2 + (a_1l_1+l_2)x^2v,$$

which implies $l_1 = l_2 = 0$.

Therefore, (4.10) can be rewritten as

$$l_3\tilde{x}^{2n+1}w + l_4\tilde{x}^{2n+1}vw = 0. \tag{4.11}$$

Again by (4.4), we have

$$\begin{aligned} l_3\tilde{x}^{2n+1}w + l_4\tilde{x}^{2n+1}vw &= \begin{cases} l_3x^{2n+1}w + l_4x^{2n+1}vw, & \text{if } a_1 = 0, \\ l_3x^{2n+1}(1+v)w + l_4x^{2n+1}(1+v)vw, & \text{if } a_1 = 1, \end{cases} \\ &= \begin{cases} l_3x^{2n+1}w + l_4x^{2n+1}vw, & \text{if } a_1 = 0, \\ l_3x^{2n+1}w + (l_3+l_4)x^{2n+1}vw, & \text{if } a_1 = 1. \end{cases} \end{aligned}$$

which implies $l_3 = l_4 = 0$ too.

2° For deg $= -(2k+1)$ with $1 \leq k \leq n$, the proof is similar as above. Assume that there is $(l_1, l_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that

$$l_1\tilde{x}^{2k+1} + l_2\tilde{x}^{2k+1}v = 0. \tag{4.12}$$

Then by (4.4) we get

$$\begin{aligned} 0 &= \begin{cases} l_1x^{2k+1} + l_2x^{2k+1}v, & \text{if } a_1 = 0, \\ l_1x^{2k+1}(1+v) + l_2x^{2k+1}(1+v)v, & \text{if } a_1 = 1, \end{cases} \\ &= \begin{cases} l_1x^{2k+1} + l_2x^{2k+1}v, & \text{if } a_1 = 0, \\ l_1x^{2k+1} + (l_1+l_2)x^{2k+1}v, & \text{if } a_1 = 1, \end{cases} \end{aligned}$$

which implies $l_1 = l_2 = 0$.

3° For deg $= -2k$ with $0 \leq k \leq n$, the proof follows from the linear independence of x^{2k} and $x^{2k}v$ when $k > 0$, and that of 1, v, $x^{2n}w$, and $x^{2n}vw$ when $k = 0$, due to $\tilde{x}^{2k} = x^{2k}$ by (4.4). Then the claim is proved.

Step 2: Construction of $\tilde{v}$.

Again by Proposition 4.1, each of $\mathbb{H}_{-(2n+1)}(L_eM;\mathbb{Z}_2)$ and $\mathbb{H}_{-(2n+1)}(L_gM;\mathbb{Z}_2)$ has a generator in $\{\tilde{x}^{2n+1}, \tilde{x}^{2n+1}v, \tilde{x}^{2n+1}(1+v)\}$. Since

$$\tilde{x}^{2n+1} \in \mathbb{H}_{-(2n+1)}(M;\mathbb{Z}_2) \subseteq \mathbb{H}_{-(2n+1)}(L_eM;\mathbb{Z}_2),$$

either $\tilde{x}^{2n+1}v$ or $\tilde{x}^{2n+1}(1+v)$ belongs to $\mathbb{H}_{-(2n+1)}(L_gM;\mathbb{Z}_2)$. According to the definition of the loop product (cf. pp. 6-7, in [12]), we have

$$\begin{aligned} \mathbb{H}_*(L_eM;\mathbb{Z}_2) \bullet \mathbb{H}_*(L_eM;\mathbb{Z}_2) &\subset \mathbb{H}_*(L_eM;\mathbb{Z}_2), & (4.13) \\ \mathbb{H}_*(L_eM;\mathbb{Z}_2) \bullet \mathbb{H}_*(L_gM;\mathbb{Z}_2) &\subset \mathbb{H}_*(L_gM;\mathbb{Z}_2), & (4.14) \\ \mathbb{H}_*(L_gM;\mathbb{Z}_2) \bullet \mathbb{H}_*(L_gM;\mathbb{Z}_2) &\subset \mathbb{H}_*(L_eM;\mathbb{Z}_2). & (4.15) \end{aligned}$$

So there exists

$$\tilde{v} \in \mathbb{H}_0(L_gM;\mathbb{Z}_2) \subset \mathbb{H}_0(LM;\mathbb{Z}_2) = \mathrm{Span}\{1, v, \tilde{x}^{2n}w, \tilde{x}^{2n}vw\}$$

such that

$$0 \neq \tilde{x}^{2n+1}\tilde{v} \in \mathbb{H}_{-(2n+1)}(L_gM;\mathbb{Z}_2),$$

and we assume

$$\tilde{v} = b_0v + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}vw,$$

with $b_0, b_1, b_2, b_3 \in \mathbb{Z}_2$.

Now we claim $b_0 = 1$, that is

$$\tilde{v} = v + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}vw. \tag{4.16}$$

If not, then we get

$$\tilde{x}^{2n+1}\tilde{v} = \tilde{x}^{2n+1}(b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}vw) = b_1\tilde{x}^{2n+1} \in \mathbb{H}_{-(2n+1)}(L_eM;\mathbb{Z}_2),$$

a contradiction.

It then follows from (4.16) that

$$\tilde{v}^2 = \begin{cases} v^2, & \text{if } b_1 = 0, \\ v^2 + 1, & \text{if } b_1 = 1, \end{cases} \tag{4.17}$$

and

$$\begin{aligned} v &= \tilde{v} + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}vw \\ &= \tilde{v} + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n}(\tilde{v} + b_1 + b_2\tilde{x}^{2n}w + b_3\tilde{x}^{2n})w \\ &= \tilde{v} + b_1 + (b_2 + b_1b_3)\tilde{x}^{2n}w + b_3\tilde{x}^{2n}\tilde{v}w. \end{aligned} \tag{4.18}$$

That is, v can be represented in terms of $\tilde{x}$, $\tilde{v}$ and w.

Now by (4.16), (4.17) and (4.18), similar arguments as those in Step 1 then yield

$$\mathbb{Z}_2[\tilde{x}, v, w]/(\tilde{x}^{2n+2}, v^2 - (n+1)w\tilde{x}^{2n}) \cong \mathbb{Z}_2[\tilde{x}, \tilde{v}, w]/(\tilde{x}^{2n+2}, \tilde{v}^2 + b_1 - (n+1)w\tilde{x}^{2n}). \tag{4.19}$$

Step 3: Construction of $\tilde{w}$.

We consider

$$w \in \mathbb{H}_{2n}(LM;\mathbb{Z}_2) = \mathrm{Span}\{w, \tilde{v}w, \tilde{x}^{2n}w^2, \tilde{x}^{2n}\tilde{v}w^2\}.$$

If

$$w \in \mathbb{H}_{2n}(L_eM;\mathbb{Z}_2) \cup \mathbb{H}_{2n}(L_gM;\mathbb{Z}_2), \tag{4.20}$$

we need do nothing but define $\tilde{w} = w$.

While if

$$w \in \mathbb{H}_{2n}(L_eM;\mathbb{Z}_2) + \mathbb{H}_{2n}(L_gM;\mathbb{Z}_2),$$

then due to the linear independence of w with the other three generators, there exist $c_1, c_2, c_3 \in \mathbb{Z}_2$ so that w can be decomposed into two parts as follows

$$w = (w + c_1\tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2) + (c_1\tilde{v}w^2 + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2),$$

with one part in $\mathbb{H}_{2n}(L_eM;\mathbb{Z}_2)$ and the other in $\mathbb{H}_{2n}(L_gM;\mathbb{Z}_2)$.

We continue the discussion in three cases according to the possible values of c_1, c_2 and c_3.

(1) If $c_1 = 0$, we define

$$\begin{aligned} \tilde{w} &= w + c_1\tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &= w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2. \end{aligned} \tag{4.21}$$

Then, multiplying by $\tilde{x}^{2n}$ both sides of (4.21) we get

$$\tilde{x}^{2n}\tilde{w} = \tilde{x}^{2n}w \tag{4.22}$$

and so

$$\begin{aligned} w &= \tilde{w} + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &= \tilde{w} + c_2\tilde{x}^{2n}\tilde{w}w + c_3\tilde{x}^{2n}\tilde{v}\tilde{w}w \\ &= \tilde{w} + c_2\tilde{x}^{2n}\tilde{w}^2 + c_3\tilde{x}^{2n}\tilde{v}\tilde{w}^2 \end{aligned} \tag{4.23}$$

That is, w can be represented in terms of $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$.

By (4.21), (4.22) and (4.23), similar arguments as those in Step 1 then yield

$$\mathbb{Z}_2[\tilde{x},\tilde{v},w]/(\tilde{x}^{2n+2},\tilde{v}^2 + b_1 - (n+1)w\tilde{x}^{2n}) \cong \mathbb{Z}_2[\tilde{x},\tilde{v},\tilde{w}]/(\tilde{x}^{2n+2},\tilde{v}^2 + b_1 - (n+1)\tilde{w}\tilde{x}^{2n}). \tag{4.24}$$

(2) If $c_1 = 1$ and $b_1 = 0$, we also define

$$\begin{aligned} \tilde{w} &= w + c_1\tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &= w(1+\tilde{v}) + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2. \end{aligned} \tag{4.25}$$

Then, multiplying both sides of (4.25) by $\tilde{x}^{2n}(1+\tilde{v})$ and $(1+\tilde{v})$ respectively we get

$$\tilde{x}^{2n}\tilde{w}(1+\tilde{v}) = \tilde{x}^{2n}w(1+\tilde{v})^2 = \tilde{x}^{2n}w, \tag{4.26}$$

and

$$\tilde{w}(1+\tilde{v}) = w(1+\tilde{v}^2) + c_2\tilde{x}^{2n}(1+\tilde{v})w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2, \tag{4.27}$$

where we have used the identities

$$(1+\tilde{v})^2 = 1 + 2\tilde{v} + \tilde{v}^2 = 1 + \tilde{v}^2 = 1 + v^2 = 1 + (n+1)\tilde{x}^{2n}w.$$

Notice also that n must be odd in this case. Indeed, if n is even, $\tilde{x}^{2n}\tilde{w}(1+\tilde{v}) = \tilde{x}^{2n}w = \tilde{v}^2 \in \mathbb{H}_0(L_eM;\mathbb{Z}_2)$ by (4.26), but both $\tilde{x}^{2n}\tilde{w}$ and $\tilde{x}^{2n}\tilde{w}\tilde{v}$ are nonzero and belong to different parts of $\mathbb{H}_0(L_eM;\mathbb{Z}_2) \cup \mathbb{H}_0(L_gM;\mathbb{Z}_2)$, a contradiction. As a result, we have

$$\tilde{v}^2 = (n+1)\tilde{x}^{2n}w = 0. \tag{4.28}$$

It then follows by (4.26), (4.27) and (4.28) that

$$\begin{aligned} w &= \tilde{w}(1+\tilde{v}) + w\tilde{v}^2 + c_2\tilde{x}^{2n}(1+\tilde{v})w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &= \tilde{w}(1+\tilde{v}) + c_2\tilde{x}^{2n}(1+\tilde{v})\tilde{w}^2 + c_3\tilde{x}^{2n}\tilde{v}\tilde{w}^2. \end{aligned} \tag{4.29}$$

That is, w can be represented in terms of $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$.

By (4.25), (4.28) and (4.29), similar arguments as those in Step 1 then yield

$$\mathbb{Z}_2[\tilde{x},\tilde{v},w]/(\tilde{x}^{2n+2},\tilde{v}^2) \cong \mathbb{Z}_2[\tilde{x},\tilde{v},\tilde{w}]/(\tilde{x}^{2n+2},\tilde{v}^2). \tag{4.30}$$

(3) If $c_1 = 1$ and $b_1 = 1$, we define

$$\begin{aligned} \tilde{w} &= c_1\tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &= \tilde{v}w + c_2\tilde{x}^{2n}w^2 + c_3\tilde{x}^{2n}\tilde{v}w^2 \\ &. \end{aligned} \tag{4.31}$$

Multiplying both sides of (4.31) by $\tilde{x}^{2n}\tilde{v}$ and $\tilde{v}$, we get

$$\tilde{x}^{2n}\tilde{w}\tilde{v} = \tilde{x}^{2n}w\tilde{v}^2 = \tilde{x}^{2n}w, \tag{4.32}$$

and

$$\begin{aligned} \tilde{w}\tilde{v} &= \tilde{v}^2w + c_2\tilde{x}^{2n}\tilde{v}w^2 + c_3\tilde{x}^{2n}\tilde{v}^2w^2 \\ &= (\tilde{x}^{2n}w+1)w + c_2\tilde{x}^{2n}\tilde{v}w^2 + c_3\tilde{x}^{2n}\tilde{v}^2w^2 \\ &= w + \tilde{x}^{2n}w^2 + c_2\tilde{x}^{2n}\tilde{v}w^2 + c_3\tilde{x}^{2n}w^2 \end{aligned} \tag{4.33}$$

By (4.32) and (4.33) we have

$$\begin{aligned} w &= \tilde{w}\tilde{v} + c_2\tilde{x}^{2n}\tilde{v}w^2 + (1+c_3)\tilde{x}^{2n}w^2 \\ &= \tilde{w}\tilde{v} + c_2\tilde{x}^{2n}\tilde{v}^3\tilde{w}^2 + (1+c_3)\tilde{x}^{2n}\tilde{v}^2\tilde{w}^2 \\ &= \tilde{w}\tilde{v} + c_2\tilde{x}^{2n}\tilde{v}\tilde{w}^2 + (1+c_3)\tilde{x}^{2n}\tilde{w}^2. \end{aligned} \tag{4.34}$$

That is, w can be represented in terms of $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$.

Then by (4.31), (4.32) and (4.34), similar arguments as those in Step 1 yield

$$\mathbb{Z}_2[\tilde{x},\tilde{v},w]/(\tilde{x}^{2n+2},\tilde{v}^2+1-(n+1)w\tilde{x}^{2n}) \cong \mathbb{Z}_2[\tilde{x},\tilde{v},\tilde{w}]/(\tilde{x}^{2n+2},\tilde{v}^2+1-(n+1)\tilde{x}^{2n}\tilde{v}\tilde{w}). \tag{4.35}$$

Finally, summarizing (4.9), (4.19), (4.24), (4.30) and (4.35) together, we obtain (4.2) and complete the proof of the lemma. $\square$

Remark 4.1 *In the sequel, we are not concerned with the precise relationship of $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$ in Lemma 4.1. Actually whatever it is, $\mathbb{H}_*(LM;\mathbb{Z}_2)$ is generated by*

$$\left\{\tilde{x}^l\tilde{v}^m\tilde{w}^n \mid 0 \le l \le 2n+1, 0 \le m \le 1 \text{ and } 0 \le n < +\infty\right\},$$

and we only use $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$ to separate $\mathbb{H}_*(L_eM;\mathbb{Z}_2)$ and $\mathbb{H}_*(L_gM;\mathbb{Z}_2)$ from $\mathbb{H}_*(LM;\mathbb{Z}_2)$ in Theorem 4.1. For notational simplicity, we still use x, v and w later instead of $\tilde{x}$, $\tilde{v}$ and $\tilde{w}$ respectively.

We need the following lemma for the purpose of computations.

Lemma 4.2 *For every $x, y \in \mathbb{H}_*(LM;\mathbb{Z}_2)$ and $k \in \mathbb{N}$,*

(i) $\triangle(x^2) = 0$, $\{x, y^2\} = 0$.

(ii) $\triangle(x^{2k}) = 0$, $\{x, y^{2k}\} = 0$.

(iii) $\triangle(xy^{2k}) = \triangle(x)y^{2k}$ *and* $\{x, y^{2k+1}\} = \{x, y\}y^{2k}$.

Proof: (i) According to the definition 4.1 in [12] (p. 12) and observing that the coefficient field is $\mathbb{Z}_2$, we have $\{x, x\} = 0$. By the B-V formulae, $\triangle(x^2) = x \bullet \triangle(x) + \triangle(x) \bullet x + \{x, x\} = 0$. Similarly,

$$\{x, y^2\} = \{x, y\} \bullet y + y \bullet \{x, y\} = 0.$$

(ii) $\triangle(x^{2k}) = \triangle((x^k)^2) = 0$, $\{x, y^{2k}\} = \{x, (y^k)^2\} = 0$.

(iii) By (ii),

$$\triangle(xy^{2k}) = \triangle(x) \bullet y^{2k} + x \bullet \triangle(y^{2k}) + \{x, y^{2k}\} = \triangle(x)y^{2k}.$$

Similarly,

$$\{x, y^{2k+1}\} = \{x, y \bullet y^{2k}\} = \{x, y\} \bullet y^{2k} + y \bullet \{x, y^{2k}\} = \{x, y\}y^{2k}.$$

□

In the sequel, we will focus on the computation of the third pages of $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ whose second pages are $H_*(BS^1) \otimes \mathbb{H}_*(L_eM)$ and $H_*(BS^1) \otimes \mathbb{H}_*(L_gM)$, respectively. As usual, we introduce the following definition for convenience of writing.

Definition 4.1 *For a subset S of $H_*(BS^1) \otimes \mathbb{H}_*(L_eM)$ (resp. $H_*(BS^1) \otimes \mathbb{H}_*(L_eM)$), the set generated by S consists of linear combinations of elements of the same topological degree in S with coefficients in $\mathbb{Z}_2$.*

Lemma 4.3 *For $M = \mathbb{R}P^{2n+1}$ with $n \ge 1$, $\triangle \equiv 0$ on $\mathbb{H}_*(L_eM;\mathbb{Z}_2)$.*

Proof: Assume by contradiction that there exists $y \in \mathbb{H}_*(L_eM;\mathbb{Z}_2)$ such that $\triangle(y) \neq 0$. Then for every $k \in \mathbb{N}\cup\{0\}$, $yw^{2k} \in \mathbb{H}_*(L_eM;\mathbb{Z}_2)$ and by Lemma 4.2 $\triangle(yw^{2k}) = \triangle(y)w^{2k} \neq 0$. According to Lemma 2.2, for the Leray-Serre spectral sequence $\mathcal{LS}(L_eM)$ we have

$$d_2(u^p \otimes yw^{2k}) = u^{p-1} \otimes \triangle(yw^{2k}) \neq 0,\ \forall\ p \in \mathbb{N}.$$

Thus at least the elements generated by the set

$$\{u^p \otimes yw^{2k} \mid k \in \mathbb{N}\cup\{0\},\ p \in \mathbb{N}\}$$

are killed when the spectral sequence passes to the third page from the second one, since they are not in the kernel of d_2. As a result, the Poincaré series $P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t)$ associated to the third page of $\mathcal{LS}(L_eM)$ satisfies

$$\begin{aligned} P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t) &\leq P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t) - t^{2n+1}t^{|y|}\sum_{p=1}^{+\infty}\sum_{k=0}^{+\infty} t^{2p}t^{4kn} \\ &= P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t) - \frac{t^{2n+3+|y|}}{(1-t^2)(1-t^{4n})}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} P^{S^1}(L_gM;\mathbb{Z}_2)(t) &= P^{S^1}(LM;\mathbb{Z}_2)(t) - P^{S^1}(L_eM;\mathbb{Z}_2)(t) \\ &\geq P^{S^1}(LM;\mathbb{Z}_2)(t) - P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t) \\ &\geq P^{S^1}(LM;\mathbb{Z}_2)(t) - P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t) + \frac{t^{2n+3+|y|}}{(1-t^2)(1-t^{4n})} \\ &= \frac{1-t^{2n+2}}{(1-t^{2n})(1-t^2)} + \frac{t^{2n+3+|y|}}{(1-t^2)(1-t^{4n})} \\ &> \frac{t^{2n+3+|y|}}{(1-t^2)(1-t^{4n})}, \end{aligned} \tag{4.36}$$

where to get the second equality we have used direct computations for $P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t)$ and Proposition 4.2. Since the right hand side of (4.36) contains the multiplicity factor

$$\frac{1}{(1-t^2)(1-t^{4n})},$$

it follows that the Betti number sequence of $P^{S^1}(L_gM;\mathbb{Z}_2)(t)$ is unbounded. It then yields a contradiction to Proposition 3.2 and completes the proof. □

Next we investigate the possible B-V algebraic structures of $(\mathbb{H}_*(LM;\mathbb{Z}_2),\bullet,\triangle)$.

Theorem 4.1 *Let $M = \mathbb{R}P^{2n+1}$ with $n \geq 1$, then $(\mathbb{H}_*(LM;\mathbb{Z}_2),\bullet,\triangle)$ has the following four possible B-V algebraic structures.*

(A) *If* $w \in \mathbb{H}_*(L_eM;\mathbb{Z}_2)$, *then we have*

$$\triangle(x) = 0,\ \triangle(v) = 0,\ \triangle(w) = 0, \tag{4.37}$$

and

$$\{x, v\} = v \text{ or } v + x^{2n}vw,\ \{v, w\} = 0,\ \{x, w\} = 0. \tag{4.38}$$

(B) *If* $w \in \mathbb{H}_*(L_gM;\mathbb{Z}_2)$, *then we have*

$$\triangle(x) = 0,\ \triangle(v) = 0,\ \triangle(w) = 0, \tag{4.39}$$

and

$$\{x, v\} = v,\ \{v, w\} = 0,\ \{x, w\} = w \text{ or } w + x^{2n}vw^2. \tag{4.40}$$

Proof: We carry out the proof in two steps.

Step 1. *Studies in Case (A).*

When $w \in \mathbb{H}_*(L_eM;\mathbb{Z}_2)$, then $\mathbb{H}_*(L_eM;\mathbb{Z}_2)$ is generated by

$$\{x^a w^c \mid 0 \leq a \leq 2n+1,\ c \geq 0\}.$$

By Lemma 4.3, we have

$$\triangle(x) = 0,\ \triangle(w) = 0 \text{ and } \triangle(xw) = 0, \tag{4.41}$$

by the B-V formulae which imply

$$\{x, w\} = \triangle(xw) - \triangle(x)w - x\triangle(w) = 0. \tag{4.42}$$

Here note that there is an alternative topological way to prove $\triangle(x) = 0$. Let X be a representative of x, i.e., $[X] = x$. Since the action $\eta : S^1 \times L_eM \to L_eM$ of (2.1) is trivial on $M \subseteq L_eM$ and $x \in \mathbb{H}_{-1}(M;\mathbb{Z}_2)$, we have $\eta(S^1 \times X) = X$ and so

$$\triangle(x) = \eta_*([S^1] \times x) = [\eta(S^1 \times X)] = [X] = 0,$$

where $[S^1]$ is the generator of $H_1(S^1;\mathbb{Z}_2)$ and the last equality is due to $\triangle(x) \in \mathbb{H}_0(L_eM;\mathbb{Z}_2)$.

Next we consider the fibration

$$L_eM \to L_eM \times_{S^1} ES^1 \to BS^1,$$

and its Leray-Serre spectral sequence $\{\mathbb{E}^r, d_r\} = \mathcal{LS}(L_eM)$.

Since $d_2(u \otimes x^{2n+1}) = \triangle(x^{2n+1}) = 0$ by Lemma 2.2, $x^{2n} \in \mathbb{H}_{-2n}(L_eM;\mathbb{Z}_2)$ survives until $\mathbb{E}^\infty$. Note that the degree of x^{2n} is $-2n$, and therefore it contributes the term t^{-2n} to $\mathbb{H}_*^{S^1}(L_eM;\mathbb{Z}_2)$.

However, since $\mathbb{H}_*^{S^1}(L_eM;\mathbb{Z}_2)$ is the desuspension of $H_*^{S^1}(L_eM;\mathbb{Z}_2)$, one should multiply it by $t^{\dim M}=t^{2n+1}$ to get a term t in $P^{S^1}(L_eM;\mathbb{Z}_2)(t)$ finally (cf. Proof of Theorem 1.1 in [35], pp. 321-322). Here note that the Poincaré series $P^{S^1}(L_eM;\mathbb{Z}_2)(t)$ computed in our paper corresponds to $H_*^{S^1}(L_eM;\mathbb{Z}_2)$ instead of $\mathbb{H}_*^{S^1}(L_eM;\mathbb{Z}_2)$.

However by Proposition 4.2, we have

$$P^{S^1}(LM;\mathbb{Z}_2)(t)=\frac{1}{1-t^{2n}}\left(1+\frac{1+t}{1-t^2}\right)\frac{1-t^{2n+2}}{1-t^2},$$

from which we know that the coefficient of t in $P^{S^1}(LM;\mathbb{Z}_2)(t)$ is 1. Since

$$P^{S^1}(LM;\mathbb{Z}_2)(t)=P^{S^1}(L_eM;\mathbb{Z}_2)(t)+P^{S^1}(L_gM;\mathbb{Z}_2)(t),$$

the Leray-Serre spectral sequence of the fiberation

$$L_gM\to L_gM\times_{S^1}ES^1\to BS^1,$$

must kill $x^{2n}v\in\mathbb{H}_{-2n}(L_gM;\mathbb{Z}_2)$, when it passes to the third page from the second one. As a result, we obtain $\triangle(x^{2n+1}v)=x^{2n}v$. Thus by Lemma 4.2, we obtain

$$0=\triangle^2(x^{2n+1}v)=\triangle(x^{2n}v)=x^{2n}\triangle(v), \tag{4.43}$$

and so

$$\begin{aligned} x^{2n}v &= \triangle(x^{2n+1}v)\\ &= \triangle(x^{2n+1})v+x^{2n+1}\triangle(v)+\{x^{2n+1},v\}\\ &= \{x^{2n+1},v\}\\ &= x^{2n}\{x,v\}. \end{aligned} \tag{4.44}$$

By (4.44), $\{x,v\}\neq 0$ and so we can assume

$$\{x,v\}=x^av^bw^c$$

with $0\le a\le 2n+1$, $0\le b\le 1$ and $c\ge 0$, or a sum of terms like this. Then we get

$$-a+2nc=|x^av^bw^c|=|\{x,v\}|=|x|+|v|+1=0,$$

i.e., $c=\frac{a}{2n}$. It then can be checked that only $(a,c)=(0,0)$ and $(a,c)=(2n,1)$ are the possible required pairs of non-negative integers for all $n\ge 1$. Observing by the B-V formulae

$$\{x,v\}=\triangle(x\bullet v)-\triangle(x)\bullet v-x\bullet\triangle(v),$$

we have $\{x,v\}\in\mathbb{H}_*(L_gM;\mathbb{Z}_2)$. Therefore, we have

$$\{x,v\}=v,\ x^{2n}vw \text{ or } v+x^{2n}vw.$$

Again by (4.44), we have

$$\{x,v\}=v \text{ or } v+x^{2n}vw. \tag{4.45}$$

Similarly if $\triangle(v)\neq 0$, we can assume that

$$\triangle(v)=x^av^bw^c$$

with $0\le a\le 2n+1$, $0\le b\le 1$ and $c\ge 0$, or a sum of terms like this. Then we get

$$-a+2nc=|x^av^bw^c|=|\triangle(v)|=|v|+1=1,$$

i.e., $c=\frac{a+1}{2n}$. It can be checked that only

$$(a,c)=\begin{cases}(2n-1,1), & \text{if } n\ge 2;\\ (1,1) \text{ or } (3,2), & \text{if } n=1,\end{cases}$$

are the possible required pairs of non-negative integers for all $n\ge 1$.

Since $\triangle:\mathbb{H}_*(L_gM;\mathbb{Z}_2)\to\mathbb{H}_*(L_gM;\mathbb{Z}_2)$, by (4.43) we have

$$\triangle(v)=\begin{cases}0 \text{ or } x^{2n-1}vw, & \text{if } n\ge 2;\\ 0 \text{ or } x^3vw^2, & \text{if } n=1.\end{cases}$$

Furthermore, we claim

$$\triangle(v)=0. \tag{4.46}$$

In fact, we prove (4.46) in two cases $n\ge 2$ and $n=1$.

When $n\ge 2$, if $\triangle(v)=x^{2n-1}vw$ for some $n\ge 2$, then by (4.45) we have

$$\begin{aligned}\triangle(xv)&=\triangle(x)v+x\triangle(v)+\{x,v\}\\&=x^{2n}vw+\{x,v\}\\&=\begin{cases}x^{2n}vw+v, & if\ \{x,v\}=v;\\ v, & if\ \{x,v\}=v+x^{2n}vw,\end{cases}\end{aligned}$$

and so

$$\begin{aligned}0=\triangle^2(xv)&=\begin{cases}\triangle(x^{2n}vw)+\triangle(v), & if\ \{x,v\}=v;\\ \triangle(v), & if\ \{x,v\}=v+x^{2n}vw,\end{cases}\\&=\begin{cases}x^{2n}\triangle(vw)+x^{2n-1}vw, & if\ \{x,v\}=v;\\ x^{2n-1}vw, & if\ \{x,v\}=v+x^{2n}vw,\end{cases}\end{aligned}$$

a contradiction.

When $n = 1$, if $\triangle(v) = x^3vw^2$, then

$$0 = \triangle^2(v) = \triangle(x^3vw^2) = \triangle(x^3v)w^2 = x^2vw^2,$$

a contradiction too, and then (4.46) is proved.

Finally, we come to prove

$$\{v, w\} = 0. \tag{4.47}$$

In fact, by arguments on topological degrees similar to the above discussion, we have

$$\{v, w\} = \begin{cases} 0 \text{ or } x^{2n-1}vw^2, & \text{if } n \geq 2; \\ 0,\ xvw^2,\ x^3vw^3 \text{ or } xvw^2 + x^3vw^3, & \text{if } n = 1. \end{cases}$$

If $\{v, w\} = x^{2n-1}vw^2$ for some $n \geq 2$ or $\{v, w\} = xvw^2 + cx^3vw^3$ with $c \in \mathbb{Z}_2$ when $n = 1$, then it follows from (4.41), (4.45) and (4.46) that

$$\begin{aligned} 0 = x^2 \triangle^2 (vw) &= \begin{cases} x^2 \triangle (x^{2n-1}vw^2), & \text{if } \{v, w\} = x^{2n-1}vw^2; \\ x^2 \triangle (xvw^2 + cx^3vw^3), & \text{if } \{v, w\} = xvw^2 + cx^3vw^3, \end{cases} \\ &= x^{2n} \triangle (xv)w^2 \\ &= x^{2n}\{x, v\}w^2 \\ &= \begin{cases} x^{2n}vw^2, & if\ \{x, v\} = v; \\ x^{2n}vw^2 + x^{4n}vw^3, & if\ \{x, v\} = v + x^{2n}vw, \end{cases} \\ &= x^{2n}vw^2, \end{aligned}$$

where the third identity we have used $x^2 \triangle (cx^3vw^3) = cx^4 \triangle (xvw^3) = 0$, a contradiction.

When $n = 1$ and $\{v, w\} = x^3vw^3$, then we have

$$\begin{aligned} 0 &= \triangle^2(vw) = \triangle(x^3vw^3) \\ &= x^2 \triangle (xvw)w^2 \\ &= x^2(\triangle(xv)w + xv \triangle (w) + \{xv, w\})w^2 \\ &= x^2(\{x, v\}w + \{xv, w\})w^2 \\ &= \begin{cases} x^2(vw + x\{v, w\})w^2, & \text{if } \{x, v\} = v; \\ x^2(vw + x^2vw^2 + x\{v, w\})w^2, & \text{if } \{x, v\} = v + x^2vw, \end{cases} \\ &= x^2vw^3, \end{aligned} \tag{4.48}$$

which yields a contradiction too, and then (4.47) is proved.

Now by (4.41), (4.42), (4.45), (4.46) and (4.47), both (4.37) and (4.38) are proved.

Step 2. *Studies in Case (B).*

In Case (B), we have $w \in \mathbb{H}_*(L_gM;\mathbb{Z}_2)$. Then $\mathbb{H}_*(L_eM,\mathbb{Z}_2)$ is generated by

$$\{x^a vw^{2k+1},\ x^a w^{2k} \mid 1 \le a \le 2n+1,\ k \ge 0\}.$$

By Lemma 4.3, we have

$$\triangle\,(x) = 0,\ \triangle(vw) = 0 \text{ and } \triangle\,(xvw) = 0. \tag{4.49}$$

From (4.49) we obtain

$$\begin{aligned} 0 = \triangle(xvw) &= \triangle(x)vw + x\,\triangle\,(vw) + \{x, vw\} \\ &= \{x, vw\} \\ &= \{x, v\}w + \{x, w\}v. \end{aligned}$$

That is,

$$\{x, v\}w = \{x, w\}v. \tag{4.50}$$

Moreover, by the arguments almost word by word as in Case (A) we get

$$\{x, v\} = v \text{ or } v + x^{2n}w \text{ and } \triangle\,(v) = 0. \tag{4.51}$$

By the same arguments on topological degrees as in Case (A), we have

$$\{x, w\} = 0,\ w,\ x^{2n}vw^2 \text{ or } w + x^{2n}vw^2, \tag{4.52}$$

and

$$\triangle\,(w) = \begin{cases} 0 \text{ or } x^{2n-1}vw^2, & \text{if } n \ge 2; \\ 0,\ xvw^2,\ x^3w^3 \text{ or } xvw^2 + x^3w^3, & \text{if } n = 1. \end{cases} \tag{4.53}$$

From (4.50), (4.51) and (4.52) it follows

$$\{x, v\} = v \text{ and } \{x, w\} = w \text{ or } w + x^{2n}vw^2. \tag{4.54}$$

We claim also

$$\triangle\,(w) = 0. \tag{4.55}$$

In fact, by (4.53) we consider two cases for $n \ge 2$ and $n = 1$.

When $n \geq 2$, if $\triangle(w) = x^{2n-1}vw^2$, then we have

$$\begin{aligned} 0 = \triangle^2(w) = \triangle(x^{2n-1}vw^2) &= x^{2n-2} \triangle (xv)w^2 \\ &= x^{2n-2}\{x,v\}w^2 \\ &= \begin{cases} x^{2n-2}vw^2, & if\ \{x,v\} = v; \\ x^{2n-2}vw^2 + x^{4n-2}w^3, & if\ \{x,v\} = v + x^{2n}w, \end{cases} \end{aligned}$$

which yields a contradiction and proves (4.55) in this case.

When $n = 1$ and $\triangle(w) = xvw^2 + cx^3w^3$ with $c \in \mathbb{Z}_2$, an argument similar to that for the case $n \geq 2$ yields also a contradiction.

When $n = 1$ and $\triangle(w) = x^3w^3$, then we have

$$\begin{aligned} 0 = \triangle^2(w) &= \triangle(x^3w^3) \\ &= x^2 \triangle (xw)w^2 \\ &= x^2(\triangle(x)w + x \triangle (w) + \{x,w\})w^2 \\ &= x^2\{x,w\}w^2, \\ &= \begin{cases} x^2w^3, & if\ \{x,w\} = w; \\ x^2w^3 + x^4vw^4, & if\ \{x,w\} = w + x^2vw^2, \end{cases} \\ &= x^2w^3, \end{aligned}$$

which yields a contradiction too, and thus (4.55) is proved too.

Now it follows immediately by (4.49), (4.51) and (4.55) that

$$\{v,w\} = \triangle(vw) - \triangle(v)w - v \triangle (w) = 0, \tag{4.56}$$

and then (4.39) and (4.40) are proved.

The proof of Theorem 4.1 is complete. □

5 Proof of Theorem 1.1

Our idea of the proof of Theorem 1.1 is simple. We compute first the Poincaré series associated to the third pages of the Leray-Serre spectral sequences $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ for each of the four possible B-V algebraic structures obtained in Theorem 4.1 respectively. Then Theorem 1.1 follows by comparing their sums with the result obtained by Westerland in [35], i.e., Proposition 4.2.

Proof of Theorem 1.1:

We carry out the proof in two Cases (A) and (B) according to Theorem 4.1.

Step 1. *Studies in Case (A) of Theorem 4.1.*

In this case, we have $w \in \mathbb{H}_*(L_eM;\mathbb{Z}_2)$, and by (4.37) and (4.38) of Theorem 4.1, we have

$$\triangle(x)=0,\ \ \triangle(v)=0 \text{ and } \ \triangle(w)=0,$$
$$\{x,v\}=v \text{ or } v+x^{2n}vw,\ \ \{x,w\}=0 \text{ and } \{v,w\}=0.$$

We continue in three sub-steps.

(i) *Now we consider the fibration*

$$L_eM \to L_eM \times_{S^1} ES^1 \to BS^1,$$

and its Leray-Serre spectral sequence $\mathcal{LS}(L_eM)$.

Since $\triangle \equiv 0$ on $\mathbb{H}_*(L_eM;\mathbb{Z}_2)$ by Lemma 4.3, we have $d_2 \equiv 0$ on $\mathbb{E}^2$ by Lemma 2.2. Therefore $\mathbb{E}^3$ is the same as $\mathbb{E}^2$ and so the Poincaré series $P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t)$ of the third page equals to that of the second page. Observing that

$$\mathbb{E}^2=\mathbb{H}_*(L_eM)\otimes H_*(BS^1)\cong \mathbb{Z}_2[x,w,u]/(x^{2n+2}),$$

with $|x|=-1$, $|w|=2n$ and $|u|=2$ given in Lemma 2.2, we get

$$\begin{aligned}
P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t) &= P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t)\\
&= t^{2n+1}\sum_{a=0}^{2n+1}\sum_{b=0}^{+\infty}\sum_{c=0}^{+\infty} t^{-a}t^{2b}t^{2nc}\\
&= t^{2n+1}\left(\frac{1-t^{-(2n+2)}}{1-t^{-1}}\right)\frac{1}{(1-t^2)(1-t^{2n})}\\
&= \frac{1}{1-t^{2n}}\left(\frac{1-t^{2n+2}}{1-t^2}\right)\frac{1+t}{1-t^2}. \qquad (5.1)
\end{aligned}$$

(ii) *We consider the fibration*

$$L_gM \to L_gM \times_{S^1} ES^1 \to BS^1,$$

and its Leray-Serre spectral sequence $\mathcal{LS}(L_gM)$.

By direct computations on $\mathbb{H}_*(L_gM;\mathbb{Z}_2)$ we obtain

$$\begin{aligned}
\triangle\,(x^{2l}vw^c) &= x^{2l}\triangle\,(vw^c)\\
&= x^{2l}(\triangle(v)w^c+v\triangle\,(w^c)+\{v,w^c\})\\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\triangle\,(x^{2l+1}vw^c) &= x^{2l}\triangle\,(xvw^c) \\
&= x^{2l}(\triangle(xv)w^c + xv\triangle\,(w^c) + \{xv, w^c\}) \\
&= x^{2l}\triangle\,(xv)w^c \\
&= x^{2l}\{x,v\}w^c \\
&= \begin{cases} x^{2l}vw^c, & \text{if } \{x,v\} = v; \\ x^{2l}vw^c + x^{2l+2n}vw^{c+1}, & \text{if } \{x,v\} = v + x^{2n}vw, \end{cases} \\
&= \begin{cases} x^{2l}vw^c, & \text{if } \{x,v\} = v; \\ x^{2l}vw^c, & \text{if } \{x,v\} = v + x^{2n}vw \text{ and } l \geq 1, \\ vw^c + x^{2n}vw^{c+1}, & \text{if } \{x,v\} = v + x^{2n}vw \text{ and } l = 0, \end{cases} \qquad (5.2)
\end{aligned}$$

for $0 \leq l \leq n$ and $0 \leq c < +\infty$.

For the case $\{x,v\} = v + x^{2n}vw$, by (5.2) we have also

$$\begin{aligned}
\triangle\,(xvw^c + x^{2n+1}vw^{c+1}) &= \triangle(xvw^c) + \triangle(x^{2n+1}vw^{c+1}) \\
&= (vw^c + x^{2n}vw^{c+1}) + x^{2n}vw^{c+1} \qquad (5.3) \\
&= vw^c.
\end{aligned}$$

So no matter $\{x,v\} = v$ or $v + x^{2n}vw$, only the elements generated by

$$\{x^{2l+1}vw^c \mid 0 \leq l \leq n,\ c \geq 0\}$$

survive when $\mathcal{LS}(L_gM)$ passes to the third page from the second one, while the other elements generated by

$$\{x^{2l}vw^c \mid 0 \leq l \leq n,\ c \geq 0\} \cup \{u^p \otimes x^a vw^c \mid p \geq 1, 0 \leq a \leq 2n+1,\ c \geq 0\},$$

are killed since they are either in the image or not in the kernel of the second differential. As a result, we obtain

$$\begin{aligned}
P_{III}^{S^1}(L_gM;\mathbb{Z}_2)(t) &= t^{2n+1}\sum_{l=0}^{n}\sum_{c=0}^{+\infty} t^{-(2l+1)}t^{2nc} \\
&= t^{2n}\left(\frac{1-t^{-(2n+2)}}{1-t^{-2}}\right)\frac{1}{1-t^{2n}} \\
&= \frac{1}{1-t^{2n}}\left(\frac{1-t^{2n+2}}{1-t^2}\right). \qquad (5.4)
\end{aligned}$$

(iii) *Conclusion on* $P^{S^1}(L_gM;\mathbb{Z}_2)(t)$.

Comparing the sum of the two Poincaré series (5.1) and (5.4) with $P^{S^1}(LM;\mathbb{Z}_2)(t)$ given by Proposition 4.2, we have

$$P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t)+P^{S^1}_{III}(L_gM;\mathbb{Z}_2)(t)=P^{S^1}(LM;\mathbb{Z}_2)(t),$$

which implies that $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ collapse at the second and third pages respectively. Thus finally we obtain

$$P^{S^1}(L_gM;\mathbb{Z}_2)(t)=P^{S^1}_{III}(L_gM;\mathbb{Z}_2)(t)=\frac{1}{1-t^{2n}}\left(\frac{1-t^{2n+2}}{1-t^2}\right),$$

which is precisely (1.11) of Theorem 1.1 in Case (A).

Step 2. *Studies in Case (B) of Theorem 4.1.*

In this case, $w\in\mathbb{H}_*(L_gM;\mathbb{Z}_2)$. Then by (4.39) and (4.40) in Theorem 4.1 we have

$$\begin{aligned}&\triangle(x)=0,\ \triangle(v)=0 \text{ and } \triangle(w)=0,\\ &\{x,v\}=v,\ \{x,w\}=w \text{ or } w+x^{2n}vw^2 \text{ and } \{v,w\}=0.\end{aligned}$$

By similar arguments as in Step 1, we get

$$\begin{aligned}P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t) &= P^{S^1}_{II}(L_eM;\mathbb{Z}_2)(t)\\ &= \frac{1}{1-t^{2n}}\left(\frac{1-t^{2n+2}}{1-t^2}\right)\frac{1+t}{1-t^2}.\end{aligned}$$

On the other hand, by direct computations on $\mathbb{H}_*(L_gM;\mathbb{Z}_2)$ we obtain

(i) if $\{x,w\}=w$, then

$$\triangle(x^av^bw^c)=\begin{cases}0, & \text{if } a \text{ is even;}\\ x^{a-1}v^bw^c, & \text{if } a \text{ is odd,}\end{cases}$$

and

(ii) if $\{x,w\}=w+x^{2n}vw^2$, then

$$\triangle(x^av^bw^c)=\begin{cases}0, & \text{if } a \text{ is even;}\\ x^{a-1}vw^c, & \text{if } a \text{ is odd and } c \text{ is even;}\\ x^{a-1}w^c+x^{a-1+2n}vw^{c+1}, & \text{if } a \text{ is odd and } c \text{ is odd,}\end{cases}$$

where

$$b=\begin{cases}1, & \text{when } c \text{ is even;}\\ 0, & \text{when } c \text{ is odd.}\end{cases}$$

For the latter case, we also have

$$
\begin{aligned}
\triangle\left(xw^c + x^{2n+1}vw^{c+1}\right) &= \triangle(xw^c) + \triangle(x^{2n+1}vw^{c+1}) \\
&= (w^c + x^{2n}vw^{c+1}) + x^{2n}vw^{c+1} \\
&= w^c,
\end{aligned}
$$

for odd c.

Thus no matter $\{x, w\} = w$ or $w + x^{2n}vw$, only the elements generated by

$$\{x^{2l+1}w^{2k+1} \mid 0 \le l \le n,\ k \ge 0\},$$

and

$$\{x^{2l+1}vw^{2k} \mid 0 \le l \le n, k \ge 0\},$$

survive when $\mathcal{LS}(L_gM)$ passes to the third page from the second one. As a result, we obtain

$$
\begin{aligned}
P^{S^1}_{III}(L_gM;\mathbb{Z}_2)(t) &= t^{2n+1}\sum_{l=0}^{n}\sum_{k=0}^{+\infty} t^{-2l-1}t^{4kn+2n} + t^{2n+1}\sum_{l=0}^{n}\sum_{k=0}^{+\infty} t^{-2l-1}t^{4kn} \\
&= t^{2n+1}\sum_{l=0}^{n}\sum_{k=0}^{+\infty} t^{-2l-1}t^{2kn} \\
&= \frac{1}{1-t^{2n}}\left(\frac{1-t^{2n+2}}{1-t^2}\right).
\end{aligned}
$$

The rest proof is then word by word as that in Step 1.

Finally using (1.11) of $P^{S^1}(\Lambda_gM;\mathbb{Z}_2)(t)$ to find the coefficients $\bar{\beta}_i$s, by direct computations we obtain $\bar{B}_g(M) = (n+1)/(2n)$, i.e., (1.12) holds, and complete the proof of the Theorem 1.1. □

Remark 5.1 *Theorem 1.1 is not a trivial conclusion of Lemma 4.3 (or Theorem 4.1). In fact, if the sum of $P^{S^1}_{III}(L_eM;\mathbb{Z}_2)(t)$ and $P^{S^1}_{III}(L_gM;\mathbb{Z}_2)(t)$ was greater than $P^{S^1}(LM;\mathbb{Z}_2)(t)$ under one of the four possible B-V algebraic structures, one can not claim by Proposition 3.2 that the same conclusion of Lemma 4.3 holds for the higher even differentials of $\mathcal{LS}(L_eM)$. The reason is that we do not know whether the higher even differentials of $\mathcal{LS}(L_eM)$ have the "homogeneous" property possessed by the second one (Lemma 2.2), i.e., either it keeps the associated page stable or kills a "large" series, whose Betti number sequence is unbounded (cf. the proof of Lemma 4.3).*

But it is then difficult to know at which pages $\mathcal{LS}(L_eM)$ and $\mathcal{LS}(L_gM)$ collapse because the higher even differentials of the spectral sequences go mysteriously, provided that the above mentioned phenomenon happened. As a result, we could not obtain $P^{S^1}(L_gM;\mathbb{Z}_2)(t)$ any more.

6 Proof of Theorem 1.2

In this section, we apply Theorem 1.1 to obtain the resonance identity of non-contractible prime homologically visible prime closed geodesics on a Finsler $M = (\mathbb{R}P^{2n+1}, F)$ claimed in Theorem 1.2, provided the number of all the distinct prime closed geodesics on M is finite.

The proof of Theorem 1.2. Recall that we denote the homologically visible prime closed geodesics by $\mathrm{CG}_{\mathrm{hv}}(M) = \{c_1, \ldots, c_r\}$ for some integer $r > 0$ when the number of distinct prime closed geodesics on M is finite. Note also that by Lemma 3.4 we have $\hat{i}(c_j) > 0$ for all $1 \le j \le r$.

Let $w_h = M_h(\Lambda_g M)$ defined by (3.10). The Morse series of $\Lambda_g M$ is defined by

$$M(t) = \sum_{h=0}^{+\infty} w_h t^h. \tag{6.1}$$

Note that $\{w_h\}$ is a bounded sequence by the second inequality of (3.11). We now use the method in the proof of Theorem 5.4 of [27] to estimate

$$M^q(-1) = \sum_{h=0}^{q} w_h (-1)^h.$$

By (6.1) and (1.17) we obtain

$$\begin{aligned} M^q(-1) &= \sum_{h=0}^{q} w_h (-1)^h \\ &= \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{4n} \sum_{h=0}^{q} (-1)^h k_l(c_j^{2m-1}) \ ^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid h - i(c_j^{2m-1+sn_j}) = l\right\} \\ &= \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{4n} (-1)^{l+i(c_j)} k_l(c_j^{2m-1}) \ ^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \le q\right\}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} &^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \le q\right\} \\ &\quad = \ ^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \le q,\ |i(c_j^{2m-1+sn_j}) - (2m-1+sn_j)\hat{i}(c_j)| \le 2n\right\} \\ &\quad \le \ ^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid 0 \le (2m-1+sn_j)\hat{i}(c_j) \le q - l + 2n\right\} \\ &\quad = \ ^{\#}\left\{s \in \mathbb{N} \cup \{0\} \mid 0 \le s \le \frac{q - l + 2n - (2m-1)\hat{i}(c_j)}{n_j \hat{i}(c_j)}\right\} \\ &\quad \le \frac{q - l + 2n}{n_j \hat{i}(c_j)} + 1. \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\#\left\{s\in\mathbb{N}\cup\{0\}\mid l+i(c_j^{2m-1+sn_j})\le q\right\}\\
&\quad=\ \#\left\{s\in\mathbb{N}\cup\{0\}\mid l+i(c_j^{2m-1+sn_j})\le q,\ |i(c_j^{2m-1+sn_j})-(2m-1+sn_j)\hat{i}(c_j)|\le 2n\right\}\\
&\quad\ge\ \#\left\{s\in\mathbb{N}\cup\{0\}\mid i(c_j^{2m-1+sn_j})\le(2m-1+sn_j)\hat{i}(c_j)+2n\le q-l\right\}\\
&\quad\ge\ \#\left\{s\in\mathbb{N}\cup\{0\}\mid 0\le s\le\frac{q-l-2n-(2m-1)\hat{i}(c_j)}{n_j\hat{i}(c_j)}\right\}\\
&\quad\ge\ \frac{q-l-2n}{n_j\hat{i}(c_j)}-1.
\end{aligned}
$$

Thus we obtain

$$\lim_{q\to+\infty}\frac{1}{q}M^q(-1)=\sum_{j=1}^{r}\sum_{m=1}^{n_j/2}\sum_{l=0}^{4n}(-1)^{l+i(c_j)}k_l(c_j^{2m-1})\frac{1}{n_j\hat{i}(c_j)}=\sum_{j=1}^{r}\frac{\hat{\chi}(c_j)}{\hat{i}(c_j)}.$$

Since w_h is bounded, we then obtain

$$\lim_{q\to+\infty}\frac{1}{q}M^q(-1)=\lim_{q\to+\infty}\frac{1}{q}P^{S^1,q}(\Lambda_gM;\mathbb{Z}_2)(-1)=\lim_{q\to+\infty}\frac{1}{q}\sum_{k=0}^{q}(-1)^k\bar{\beta}_k(\Lambda_gM)=\bar{B}(\Lambda_gM),$$

where $P^{S^1,q}(\Lambda_gM;\mathbb{Z}_2)(t)$ is the truncated polynomial of $P^{S^1}(\Lambda_gM;\mathbb{Z}_2)(t)$ with terms of degree less than or equal to q. Thus we get

$$\sum_{j=1}^{r}\frac{\hat{\chi}(c_j)}{\hat{i}(c_j)}=\frac{n+1}{2n},$$

which proves Theorem 1.2. □

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